

STUDENTS' CONCEPTIONS OF TRIGONOMETRIC FUNCTIONS AND POSITIONING
PRACTICES DURING PAIR WORK WITH ETOYS

BY

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DISSERTATION

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ABSTRACT

This dissertation is an examination of how students learn mathematics when interacting with peers and using a computer-programming environment. Students' use of technology tools in mathematics classrooms raises important questions of how students' mathematical thinking and learning is shaped by those tools. At the same time, students' learning is shaped by how they interact with peers and make sense of mathematics together. To answer questions about the intersection of students' work with peers and use of technology tools, I conducted a study of several pairs of students working with a programming environment called Etoys on a problem about sine and cosine functions.

The dissertation is situated around three interrelated strands of work. First, I use the cKc model of *conceptions* to examine students' learning about sine and cosine functions. By combining the conceptions framework with the theory of *instrumented activity*, I consider specifically how students integrated the tools of Etoys into their mathematical thinking. In the second strand, I combine the conceptions analysis with quantitative measures of student learning gained through pre- and post-tests, illustrating how standard measures of learning can be complemented and informed through an analysis of cases. Finally, I use Systemic Functional Linguistics to extend Hiebert and Grouws's construct of *productive struggle*, to consider students' collaborative efforts towards solving a problem. Methodologically, I illustrate how the cKc framework and resources from Systemic Functional Linguistics can operationalize the constructs of students' conceptions and, respectively, collaboration. These strands of work offer views of students' learning through the lenses of their thinking about a specific concept as well as their participation in collaborative problem solving.

My findings include a description of conceptions of sine and cosine functions that students invoked through their work. More importantly, I have identified ways in which students' use of the tools in Etoys supported them to develop increasingly sophisticated conceptions. Students appropriated the tools of Etoys in different ways, and some students were able to transfer their use of Etoys to their work on a new problem. Regarding students' positioning practices, I found that when students challenged one another, they created opportunities for *collaborative productive struggle*, an activity of collaboration among students leading to positive problem solving outcomes. These findings constitute a step towards understanding when and how students can challenge one another in ways that support productive collaboration. My findings indicate that students appropriate technology tools in different ways, suggesting that tasks with technology should be designed specifically to provoke students to move beyond overly simplistic conceptions in mathematics. Implementing norms for group and pair work, especially for how students can ask questions and challenge their peers, can promote collaborative settings where students learn mathematics through collaboration around the computer.

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CHAPTER 1:

INTRODUCTION

In this dissertation I examine the nature of students' learning when working in pairs with a computer-programming environment to learn mathematics. More specifically, I study how the combination of working with peers and using the tools of a programming environment can support students to learn mathematics content and engage in collaborative problem solving practices. In recent decades, ideas of collaboration and cooperative learning have become part of the common rhetoric in mathematics education research, as is evidenced by mathematics education policy documents (National Council of Teachers of Mathematics [NCTM], 2000), empirical research (for a comprehensive review, see Esmonde, 2009a) and practitioner-oriented publications (e.g., Cohen, 1994a, Horn, 2010; Smith & Stein, 2011). Empirical research has provided a strong foundation for understanding the conditions that support learning through group work, across disciplines and in settings specific to mathematics education (Cohen, 1994a, 1994b; Sharan, 1980; Webb, 1989; 1991). However, there is still much left to be understood about how students' interactions shape the nature of their mathematical discussions during group and pair work.

Another area of mathematics education research specifically addresses the role of technology for learning mathematics. Technology offers a way for students to build mathematical understandings as they create and interact with a virtual world, such as a computer environment (Hoyles & Noss, 1992; Papert, 1980). Research has found that technology tools may support students in developing different, and often more sophisticated, mathematical ideas than when working with static representations (Heid & Blume, 2008; Hoyles & Noss, 1992; Noss & Hoyles, 1996; Papert, 1980). Mathematics education policy in recent years has called

for greater, and more meaningful, integration of technology into mathematics classrooms (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGAC], 2010; NCTM, 2000). There are many different technology environments that students can use to learn mathematics, and each of these environments is likely to shape students learning in a slightly different way. Assuming that technology offers new ways for students to understand mathematics, it is necessary to examine how students' learning is shaped by their uses of technology.

In this dissertation, I study students' use of one type of technology, a computer-programming environment. A computer-programming environment is one example of a microworld. A *microworld* refers to an environment in which the objects and relationships of a particular domain are made concrete (Edwards, 1991; Papert, 1980). In this case, the domain is mathematics, and microworlds allow students to interact with the objects and relationships of mathematics (e.g., geometric figures, variables, functions) in a concrete way. Computer-programming environments constitute one example of microworlds, which also include dynamic geometry environments [DGEs] and computer algebra systems [CAS]. A computer-programming environment is a microworld in which the student inputs commands with a programming language, maintaining symbolic control over the work (Healy & Hoyles, 2001). When using computer-programming environments for mathematics, students may rely too heavily on visual feedback (Edwards, 1991; Olive, 1991), or engage in only limited problem solving processes (Olive, 1991; Simmons & Cope, 1993). For that reason, it is important to build a broader understanding of how students come to understand mathematical ideas through the use of the tools available in a programming environment.

Within the current landscape of mathematics education research, there is opportunity to understand better the intersections of student collaboration and the use of technology for learning mathematics. Computer technologies can provide opportunities for students to collaborate and learn mathematics (Hoyles & Sutherland, 1989; Hoyles & Noss, 1992; Noss & Hoyles, 1996). This dissertation addresses the question of how students learn mathematics through pair work on an open-ended problem, with the use of technology tools. To answer this question, I have chosen a particular setting, specific mathematical content, and particular technology tools. The setting is a regular (i.e., non-honors) Algebra 2 course in a typical Midwestern public high school¹. The mathematical content is the topic of sine and cosine functions, and specifically how sine and cosine functions can be used to represent periodic phenomena. Finally, the technology tools are the tools offered by Etoys (www.squeakland.org), a computer-programming environment inspired by Logo and developed with a programming language called Squeak. This context provides a way to examine the intersections between students' mathematical learning, their use of a computer-programming environments, and their interactions with peers.

Overarching Framework of This Dissertation

This dissertation is embedded within two primary areas of research (see Figure 1.1). The first of these areas of research is how students learn mathematics through the use of technology tools, which builds more generally from how human activity is shaped through the use of tools (Verillón & Rabardel, 1995). The second area of research is how learning mathematics in the classroom is a social process. To account for these multiple examinations of students' learning, I use complementary theoretical perspectives. The theory of constructivism posits that learning is

¹ Tracking is a practice commonly employed in American high schools, in which students are separated into classrooms (regular versus honors), with the purpose that teachers can better focus their instruction to match students' needs (Hopkins, 2009).

a process of conceptual reorganization on the part of the individual (von Glasersfeld, 1993). In other words, students learn as they evolve in their ways of thinking. A sociocultural perspective identifies learning in terms of the extent to which an individual participates in the social practices of a given setting (Lave & Wenger, 1991). Learning is “increased participation” (Lave & Wenger, 1991, p. 91). The contrast between these two perspectives come from identifying learning as a process of individual sense making or as process of participation in activity.

There have been efforts to merge constructivist and sociocultural perspectives, with the argument that both individual sense making and social interactions play a crucial role in learning processes (e.g., Bauersfeld, 1992; Cobb, 1994; Ernest, 1991). There is some criticism of this approach, suggesting that moving between two theories of learning loses the coherence and insights offered by a single theory (Confrey, 1995; Lerman, 1996). However, Cobb (1994) argued that constructivist and sociocultural perspectives of learning are complementary and should inform one another². In mathematics education research, combining constructivist and sociocultural perspectives has proven to be a promising way to give account of students’ learning in mathematics classrooms (Cobb, Boufi, McClain, & Whitenack, 1997; Simon, 1995; Steffe & Tzur, 1994; Whitenack, Knipping, & Novinger, 2001; Yackel & Cobb, 1996; Yackel, Cobb, & Wood, 1991). These studies reveal that students engage in individual sense making as they participate in the social practices of a classroom. For this dissertation, I have chosen to combine the constructivist with the sociocultural approach to leverage the advantages of each. With a study of students’ conceptions, I can examine how students think about mathematics concepts.

² Sfard (1998) made a similar argument, that *acquisition* and *participation* metaphors should complement one another to provide a complete picture of learning. Although the terms “acquisition” and “construction” have different meanings in constructivist theory, Sfard also argued for the benefit of combining cognitive and social theoretical perspectives.

From a sociocultural perspective, I can examine more closely how students engage in mathematical problem solving with peers.

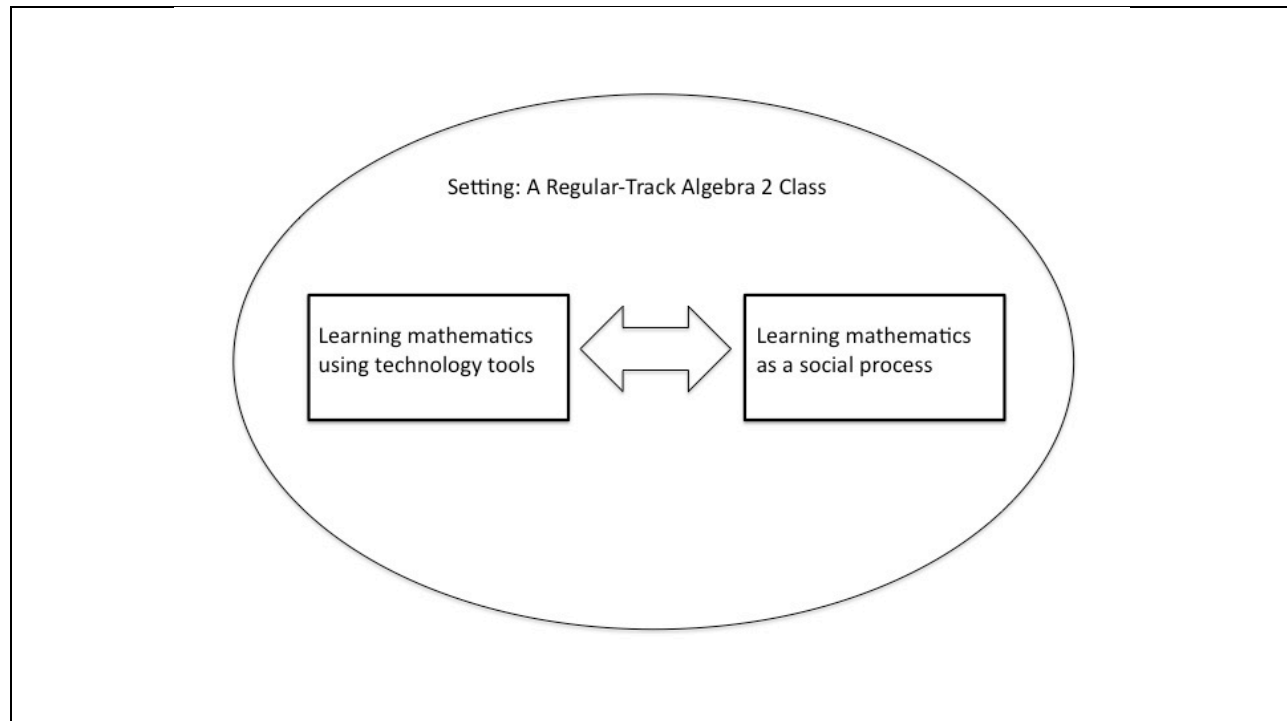


Figure 1.1. The overarching framework of the dissertation.

Learning Mathematics Using Technology Tools

To study how students learn mathematics by using technology tools, I have examined students' learning in terms of the mathematical conceptions they invoked as they worked on a problem, and how those conceptions evolved over time. Research on students' conceptions in mathematics emerged from constructivist perspectives of how students make sense of mathematical ideas (Balacheff & Gaudin, 2003; Confrey, 1990). This work has been important for mathematics education researchers to examine student thinking and the ways student thinking may diverge from standard mathematical concepts and practices (Confrey, 1990). A key theme emerging from research on students' conceptions is that students' conceptions should not be

judged according to whether they are correct or incorrect by standard mathematical thought. It is more important to note whether certain conceptions are viable for developing the solution to a problem. In constructivist theory conceptions are *viable* if they work, from the perspective of the student, for explaining a problem (von Glasersfeld, 1993). For this reason, I have examined students' learning in terms of how students' conceptions changed over the course of working on an open-ended problem about sine and cosine functions. Student learning is evidenced by students' shifts from less viable to more viable conceptions for solving the problem.

Students' mathematical thinking is shaped by the tools they use for learning mathematics. In all human activity, *tools* are artifacts designed for some specific purpose (Leontiev, 1981). Individuals use tools to perform actions for some specific purpose (Verillón & Rabardel, 1995). In the case of mathematics learning, students can use technology tools for the purpose of studying some mathematical idea or solving a problem. There are many different examples of technology tools in mathematics education research, including those offered by dynamic geometry environments (e.g., Arzarello, Olivero, Paola, & Robutti, 2002; Hollebrands, 2007; Hoyles & Noss, 1994; Laborde, 2001) and graphing calculators (e.g., Drijvers, 2000; Lesh & Doerr, 2003; Yerushalmy, 2006). Computer-programming environments offer unique ways for students to interact with mathematical ideas by using the language of a programming syntax and making connections between the inputs and outputs of a program (e.g., Clements & Battista, 1989, 1990; Edwards, 1991, 1997; Healy & Hoyles, 2001; Hoyles & Noss, 1992).

Etoys is one example of a computer-programming environment. Etoys was developed in the late 1990s, and there is still relatively little research about the use of Etoys for teaching and learning. There has been some study of using Etoys as part of problem-based learning curricula in science (e.g., Fujioka, Takada, & Hajime, 2006; Valente & Osório, 2008). In a study of

teachers' use of Etoys to promote problem solving, a group of pre-service and in-service teachers identified ways in which they could use Etoys for teaching purposes (Lee, 2012). Lundsgaard, Snit, and Blank (2013) found that, by learning some of the most basic features of the Etoys environment through a simple modeling activity, preservice science teachers began to identify ways to incorporate modeling with the use of Etoys into their own teaching. When a middle school science teacher introduced Etoys to students as a way to model the movement of the sun across the sky, students' work of programming that motion contributed to them asking further questions about how they could improve their model to better reflect the relationship between the earth and sun (Blank, Snit, & Lundsgaard, accepted). In addition, a recent project funded by the National Science Foundation has been examining the integration of Etoys into Science, Technology, Engineering, and Mathematics [STEM] curricula (Tagliarini, Narayan, & Morge, 2010). I have not identified any research that examines students' or teachers' use of Etoys specifically in mathematics settings. This dissertation makes a contribution in that direction, offering a view of how students in Algebra 2 can use Etoys to learn about sine and cosine functions.

Based on research that students' understandings of mathematics are shaped by the tools they work with (Meira, 1995), I expected that students would use the tools of Etoys in ways that would be integral to the conceptions they invoked through their work on an open-ended problem. To examine this phenomenon, I combined a framework for understanding students' conceptions (Balacheff & Gaudin, 2003) with a framework for understanding students' instrumented activity (Verillón & Rabardel, 1995). By doing so, I have been able to study students' thinking through their use of the tools in Etoys, and I have been able to examine student learning through their evolving conceptions of sine and cosine.

Learning Mathematics as a Social Process

The role of collaboration for learning mathematics is grounded in a perspective that learning is a social process, which is determined in interactions among individuals (Lave & Wenger, 1992; Vygotsky, 1978). From a sociocultural perspective, learning is defined as a process of increasing one's participation in a social activity (Lave & Wenger, 1991). In mathematics education, this means that students' learning of mathematics is defined through their increased participation in mathematical practices, including communicating, questioning, and reasoning (Greeno & MMAP, 1997). From this view, collaboration is a way for students to engage in and increase their participation in these practices. Students encounter and resolve problematic situations in mathematics through conversations with peers (Yackel, Cobb, & Wood, 1991). During collaborations, mathematical knowledge is created and taken-as-shared among members of a group (Cobb, Yackel, & Wood, 1992; Simon, 1995). The types of activities that students participate in during collaboration with peers—formulating problems, communicating ideas, and reasoning about the mathematical concepts at hand—are the activities that define students' mathematical learning.

Collaborative learning implies a “joint production of ideas,” where students listen and respond to one another, generating a shared understanding of mathematics (Staples, 2007, p. 162). Collaborative learning is not a guaranteed result of group work or pair work in mathematics classes. Students can participate in pair work, coordinating their efforts with partners, but not participate in the mathematical practices that define learning from a sociocultural perspective. However, group work and pair work create opportunities for students to engage in collaborative learning. During group work and pair work, students have opportunities for creative problem solving, for communicating ideas to others, and for using the

expertise of their peers (Cohen, 1994b). The ways that students interact with one another during group work or pair work have implications for whether students capitalize on these opportunities. To understand students' learning from a sociocultural perspective, I use a lens for understanding students' mathematical practices, and a lens for understanding how students interact with one another.

Hiebert and Grouws (2007) coined the term *productive struggle* to refer to when students “expend effort to make sense of mathematics, to figure something out that is not immediately apparent...from solving problems that are within reach” (p. 387). The construct of productive struggle helps to describe the process that occurs when students encounter, formulate, and resolve mathematics problems. The activity of productive struggle is one activity through which students learn mathematics. Specifically, through productive struggle students question and reason about mathematical ideas. When working with peers, students communicate those ideas to one another. The construct of productive struggle is especially helpful because it identifies conditions that set the stage for students to learn mathematics. To learn mathematics, students should be grappling with a problem whose solution is not obvious but is within reach. The nature of students' mathematical work is critical for students to learn mathematics through group or pair work. When students work on routine, straightforward tasks, they are not likely to learn through collaboration with peers (Cohen, 1994a, 1994b). However, when students are pushed towards productive struggle, they have more reason to engage in the mathematical practices that define learning from a sociocultural perspective.

To study students' interactions while working together, I use *positioning theory* to explain how individuals use speech and actions to position themselves in certain ways towards one another (Harré & van Langenhove, 1999). In mathematics education research, positioning

theory has shed light on how the structure of typical mathematics classrooms, and relationships between teachers, students, and text, impose certain constraints on the ways that students are able to position themselves (Herbel-Eisenmann, 2007; Herbel-Eisenmann & Wagner, 2007, 2010). Specifically, when students interact in whole-class settings, they tend to position themselves as under the mathematical authority of the teacher and curriculum. When students work together in groups, they have more agency to position themselves in positions of authority than when interacting in whole-class settings. The ways that students position themselves, for example as *experts* and *novices* (Esmonde, 2009b), have implications for how students interact with one another and with mathematical content.

I have paired the framework of students' positioning practices with the idea of productive struggle to establish a link between students' speech and actions towards one another and the mathematical outcomes of their talk. I expected that certain acts of positioning, for example asking questions or challenging their peers, would promote productive conversations about mathematical content between students. I also expected that students' use of a computer would create opportunities for students to position themselves, for example by taking control of the computer. Prior research on collaboration in mathematics education has provided quantitative links between students' behaviors during group or pair work and their mathematical outcomes (e.g., students who ask more questions during group work perform significantly better on post-lesson assessments, Webb, 1991). I have used positioning theory and productive struggle to create a qualitative link describing the connections between students' interactions and the mathematical productivity of their talk.

Connecting the Two Frameworks

With the two overarching frameworks of this dissertation, I consider two different perspectives on students' mathematical learning. First, I consider students' learning in terms of their conceptions of sine and cosine functions. Second, I consider students' learning in terms of their collaborative efforts to make sense of mathematics. Each of these considerations can be compared to students' performance on pre- and post-tests, which provide more typical measures of student learning. These two theoretical perspectives provide complementary looks at students' learning through their work on a problem with Etoys. By focusing on students' conceptions, I am able to consider how closely students' ideas are aligned with commonly accepted mathematical thought. In addition, I can pay attention to how students' use of technology tools shape their mathematical thinking. By examining students' positioning, I can consider the social processes by which students develop mathematical understanding through work on a problem. Technology tools play an important role from this perspective as well, because when students share technology tools, their use of those tools contributes to their positioning towards one another.

The Setting of the Dissertation Studies

The data for this dissertation come from the implementation of a 2-day mathematics lesson that was used in three different sections of Algebra 2 at Grove High School³ during the spring of 2013. Ms. Alexander teaches three sections of Algebra 2 at Grove High School, a school in the Midwestern United States. Grove High School has a student population of around 1000 students. Approximately 60% of students at Grove High School are White, and approximately 30% of students are Latino/a. Around 30% of the students at the school qualify

³ I use pseudonyms for all names and institutions.

for free or reduced-price lunch. Grove High School is currently on Academic Watch Status, and the school has failed to make Adequate Yearly Progress (AYP) for the past 6 years. Ms. Alexander's three sections of Algebra 2 were regular (i.e., non-honors) track classes.

The Etoys Lesson

Ms. Alexander taught a 2-day lesson about representing a periodic phenomenon with trigonometric functions in all three sections of her Algebra 2 class. The lesson, which I will refer to as the “Etoys lesson” was based on one particular problem, which I will refer to the “Ferris wheel problem.” On the Ferris wheel problem, students had to imagine that they were riding the London Eye Ferris wheel in London, England. Students had to use a Cartesian plane and write a function to represent their height off the ground at various moments in time while riding the Ferris wheel. The Etoys lesson came at the conclusion of a unit on trigonometric functions in Ms. Alexander's class. Ms. Alexander had a textbook that she used for reference (Day, Hayek, Casey, & Marks, 2004), but she did not follow the order of topics in the textbook. In her unit on trigonometric functions, Ms. Alexander saved all of the contextual problems until the end of the unit. At the time of the lesson, students in the class had studied the functions $f(x)=a\sin(bx+h)+c$ and $g(x)=a\cos(bx+h)+c$. They had discussed the aspects of amplitude, period, and vertical and horizontal shifts of these functions. However, students had not used trigonometric functions yet to represent real world phenomena. This prior knowledge was relevant for students' work on the lesson, because a central aspect of the Etoys lesson was for students to make sense of the meaning of amplitude, period, and shifts within the real-world contexts of the problem.

The mathematical content of the Etoys lesson was especially important to Ms. Alexander, because Ms. Alexander was already planning to teach a lesson about using trigonometric functions in real world applications in her Algebra 2 classes. Ms. Alexander was in the process

of aligning her curriculum with the *Common Core State Standards for Mathematics* [CCSSM], which propose that students should be able to “choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline” (National Governors Association Center for Best Practice, Council of Chief State School Officers [NGAC], 2010, p. 71). The integration of the Etoys lesson into Ms. Alexander’s usual Algebra 2 curriculum is important for understanding the role of the activity in students’ work in the class. Namely, the Etoys lesson was not seen by the teacher or students as an extra activity, or an enrichment activity outside of their normal class work. Rather, the lesson was integrated into a unit on sine and cosine functions.

Students worked in pairs⁴, at the computer, during the two days of the Etoys lesson. On the first day of the lesson, Ms. Alexander launched the problem by showing students pictures of the London Eye Ferris wheel, clarifying the goals of the lesson, and introducing students to the Etoys technology. Students worked with their partners for the remainder of the class period. On the second day of the lesson, students completed a brief warm-up and then continued working with their partners. At the conclusion of the second day of the lesson, Ms. Alexander led a discussion about the solution to the problem.

The Etoys Technology

Etoys (www.squeakland.org) is a freely available computer-programming environment. The software was inspired by Logo and developed from a programming language called Squeak. The main design principle behind Etoys is that users can create sketches, which then become objects that can be programmed. A menu of “action tiles” provides users with options for “drag-and-drop” commands within the software. For example, a user may create a sketch of a car and

⁴ There was one exception to this, with a group of three students. I will make this exception clear throughout.

then write a program to make that car drive around in a circle. Rather than directly inputting syntax with a keyboard, a user could select action tiles to make the car “go forward” by some amount and “turn” by some amount, and put those actions together in a way that will make the car drive in a circle.

The use of the Etoys software for the Ferris wheel problem was especially meaningful for two reasons. Given that one of the primary objectives of the 2-day lesson was for students to translate between a real-world context and mathematical representations, the use of a resource that would provide students with dynamic visual representations was valuable from the perspective of Ms. Alexander. In addition, the choice to use a computer-programming environment (as opposed to, for example, a dynamic geometry environment) was significant in that a programming environment allows users the control to create functions while working on the problem. Rather than manipulating objects on the screen with the use of the mouse, students had control over the syntax to program an object that would represent the motion of the Ferris wheel. Students could use the syntax in Etoys, and establish connections between the inputs and outputs of their syntax, to work on a problem about sine and cosine functions.

Research Questions

My dissertation is designed to contribute to the problem of understanding how students may learn about sine and cosine functions through pair work with the use of a computer-programming environment. My overarching research question is the following: *How do students build understandings of sine and cosine functions through working in pairs with a computer-programming environment in Algebra 2?*

To address my overarching research question, I focus on three main questions. Below, I present the three main research questions (RQs) that have guided this study. Each research

question is comprised of a set of sub-questions that serve to further articulate the details of each research question and to guide the data collection and analysis.

RQ1: How did students build understanding of sine and cosine functions in Algebra 2 through pair work on a problem with Etoys?

1.1 What conceptions of sine and cosine did students invoke throughout their work on a problem with Etoys?

1.2 How did students' conceptions of sine and cosine functions evolve through their work?

1.3 How did students use the tools of Etoys in their conceptions of sine and cosine?

RQ2: What evidence did students show of learning about sine and cosine functions through their work with Etoys?

2.1 How did students' performance on problems related to sine and cosine functions change from a pre-test to a post-test?

2.2 How did students who were lower-achieving versus higher-achieving on the pre-test compare in their changes in performance from the pre-test to the post-test?

2.3 How did students' strategies on individual tasks during a post-lesson interview compare with the strategies that they used on similar tasks during work in pairs?

RQ3: How did pairs of students' patterns of positioning support or inhibit their collective problem solving efforts?

3.1 What positions did students take up through their work with their peers during pair work at the computer?

3.2 How did students enact and change positions through their talk?

3.3 What types of positioning practices supported or hindered students in their problem solving processes during pair work?

My first two research questions, RQ1 and RQ2, are most aligned with the framework of understanding students' mathematical learning through the use of technology tools. While RQ1 gives attention to students' learning with Etoys during the in-class lesson, RQ2 addresses what mathematical understanding students gained from the Etoys lesson. My third research question, RQ3, is aligned with the framework of understanding students' learning as a social process. My three research questions serve to address the goal of examining students' learning through multiple perspectives.

The Structure of the Dissertation

With this dissertation, I answer each of my three main research questions through three distinct, though interrelated, chapters. Each of the core chapters of the dissertation, chapters 2, 3, and 4, address one of my main research questions. Chapter 2, corresponding to RQ1, is entitled "Students' Developing Conceptions of Sine and Cosine Functions Through Pair Work With Etoys." Chapter 3, corresponding to RQ2, is entitled "Student Learning About Sine and Cosine Functions Through Work on a Contextual Problem With Etoys." Chapter 4, which corresponds to RQ3, is entitled "Mathematics Students' Positioning Patterns During Pair Work With Etoys." In the concluding chapter of the dissertation, chapter 5, I summarize and integrate the main findings of each of the chapters, and I raise questions that I may pursue in future work.

Because each of chapters 2, 3, and 4 has been written as a study independent of the others, they can be read in any order. However, since each chapter builds to some degree off the previous, readers who plan to read the entire document are advised to read the chapters in order. Because each of the studies is related to the others through the theoretical framework, relevant prior literature, and setting of the study, there is some thematic repetition between the chapters in these respective sections. This repetition is justified given that each chapter was designed so that

it could be read on its own. To avoid unnecessary repetition, I have included a single list of references at the end of the dissertation.

CHAPTER 2:

STUDENTS' DEVELOPING CONCEPTIONS OF SINE AND COSINE FUNCTIONS THROUGH PAIR WORK WITH ETOYS

The ways that students learn mathematics are tightly interwoven with the nature of the problems they work on (Balacheff & Gaudin, 2002, 2003; Vergnaud, 1982, 1998) and the tools they use to solve those problems (Bartolini Bussi & Mariotti, 1999; Noss & Hoyles, 1996; Pea, 1987). In this study, I examine students' conceptions of sine and cosine functions as they worked on a problem with technology tools in an Algebra 2 course. The tools that students used were the tools available in a computer-programming environment called Etoys. Etoys is an object-oriented, drag-and-drop computer-programming environment. Etoys includes features that are designed to enable users to engage in programming activities without needing to master a programming syntax. I began my study with an expectation that students' learning about sine and cosine functions would be intertwined with their use of Etoys. With that assumption, I sought to explain precisely how students' use of Etoys would shape their conceptions of sine and cosine.

The word "conception" has a long history in mathematics education research (see Confrey, 1990), although the meaning of the word conception has often been left implicit rather than explicitly defined (Balacheff & Gaudin, 2002). Informally, *students' conceptions* refer to "categories of children's beliefs, theories, meanings, and explanations" (Confrey, 1990, p. 4). Research on students' conceptions in mathematics has shed light on how students think about mathematical concepts from early grades through secondary mathematics (Confrey, 1990). A challenge of examining students' conceptions is that it requires inferences about what students likely are thinking. I use an operational definition of conception from the cK₂ framework

(Balacheff & Gaudin, 2002, 2003, 2009). According to this framework, a conception is defined by a quadruplet. First, there is a problem, or set of problems to be solved, which require a particular mathematical concept. Second, there are operations, or things that a student would do to solve the problem. Third, there is a set of representations, which include all of the symbols, graphs, pictures, or diagrams that give account of the problem and its solution. Finally, there is a control structure, which is what allows the student to know the operations performed are correct, or the solution has been (Balacheff & Gaudin, 2002, 2003). The critical aspect of the cK ϕ framework is that conceptions are elicited by students' work on particular problems, and conceptions are identified largely through the observable actions that students perform. Moreover, *learning* is defined as a process of moving among different conceptions. I can examine students' learning about sine and cosine functions by identifying changes in their conceptions.

A guiding assumption of research on students' use of technology for learning mathematics is that students' understanding is shaped by their use of different tools. A *tool* refers to any physical artifact that has been designed for some agreed-upon purpose (Verillón & Rabardel, 1995). In mathematics education research, studying students' use of technology tools has been important as a way to examine how computer environments may change students' learning of mathematics (e.g., di Sessa, 2000; Papert, 1980; Noss & Hoyles, 1996; Zbiek, Heid, Blume, & Dick, 2007). Technology tools may change the nature of mathematics problems, and tools certainly alter the representations, operations, and control structures that students use to work on those problems. Through the lens of the cK ϕ framework, it is clear that students' use of tools could potentially change their conceptions in multiple ways.

Research Questions

The overarching research question of this paper is RQ1: *How did students build understanding of sine and cosine functions in Algebra 2 through pair work on a problem with Etoys?* I refer to sine and cosine functions together because any sine function can be represented as a horizontal translation of a cosine function, and vice versa. This broad question serves to encompass an inquiry about how students construct meaning through their work with Etoys and their conversations with their partners. The overarching question can be considered in terms of three specific research questions:

1. What conceptions of sine and cosine did students invoke throughout their work on a problem with Etoys?
2. How did students' conceptions of sine and cosine evolve through their work with their partners?
3. How did students use the tools of Etoys in their conceptions of sine and cosine?

The first two research questions aim at understanding how students' conceptions of sine and cosine functions changed through working on a problem. I provide a detailed description of students' conceptions, so that I can identify moments when students changed their conceptions of sine and cosine. The third research question gives specific attention to how students interacted with the Etoys software. I examine how students' use of particular tools available in Etoys may have promoted changes in conceptions of sine and cosine functions. Taken together, the answers to these research questions should describe the interactions between students' learning of sine and cosine functions and their use of Etoys to work on a problem.

Review of Literature on Trigonometric Functions

Research on students' understanding of trigonometric functions informs this study for two important reasons. First, students taking mathematics courses in the United States encounter sine and cosine both as ratios and as functions, at different moments in the mathematics curriculum. In addition, a variety of representations of sine and cosine are integral to the teaching and learning of trigonometric ratios and functions. While different courses (e.g., Geometry versus Algebra) and different problems (e.g., computing the sine of an angle versus graphing a sine function) call for different representations, the ways that students use representations promote very different understanding of sine and cosine functions.

Trigonometry is one of the first topics in the mathematics curriculum in the United States to combine ideas from geometry, algebra, and graphical reasoning (NGAC, 2010; Weber, 2005). Students encounter trigonometric ratios and functions in multiple settings that rely on distinct representations (Figure 2.1). In most United States curricula, the first setting where students study trigonometry is in the case of “right triangle trigonometry,” where trigonometric functions such as sine and cosine are defined in terms of side ratios in right triangles (NGAC, 2010, p. 77). Students learn to compute sines, cosines, and tangents given certain side lengths of right triangles (Figure 2.1a). By current geometry standards, students should begin to apply formulas such as the law of sines and law of cosines (NGAC, 2010). However, at this point in the curriculum trigonometric functions are not treated as *functions*, but only as *ratios*. Students reencounter trigonometric functions later in the curriculum in the context of the unit circle (Figure 2.1b), and the unit circle gives rise to defining trigonometric functions that take all numbers as inputs (Figure 2.1c). The unit circle takes focus away from right triangles and towards using angles as inputs. The Cartesian plane represents trigonometric functions where the independent variable

represents an angle measure in radians, and the dependent variable represents the sine, cosine, or tangent of the angle.

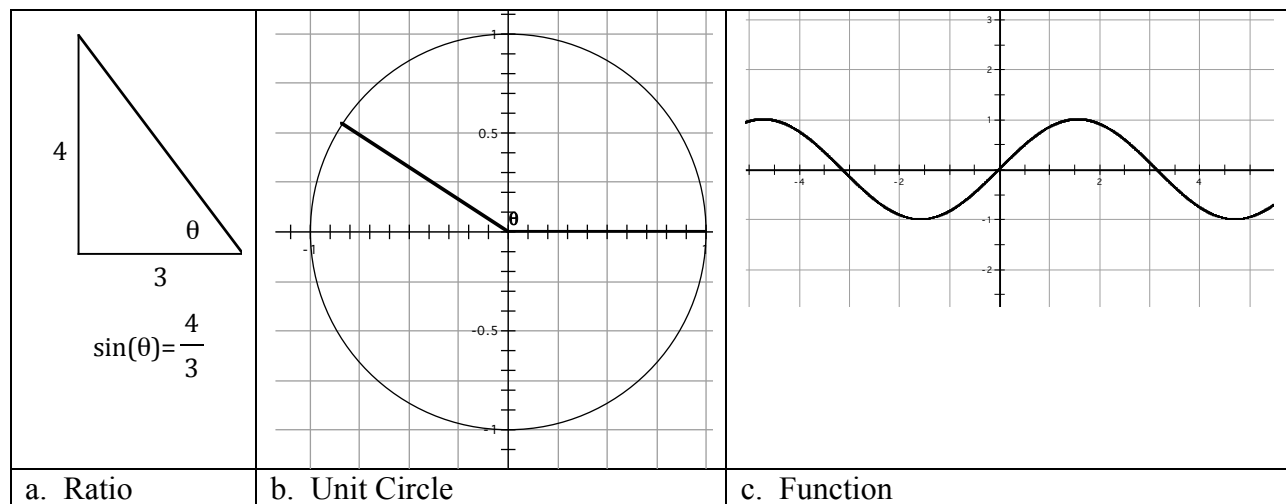


Figure 2.1. Sine and cosine encountered in different settings in the mathematics curriculum.

The challenges that students have making sense of trigonometric functions have been well documented. Chief among these seems to be that trigonometric functions such as sine and cosine cannot be expressed as formulas or algebraic procedures, and therefore students have an especially difficult time reasoning about these functions (Breidenbach, Dubinsky, Hawks, & Nichols, 1992). To understand trigonometric ratios in the context of right triangle trigonometry, students need to relate diagrams and numerical relationships and to become competent at the symbol manipulation necessary to represent those relationships (Blackett & Tall, 1991). Working in between those different representations can be a hurdle for students, even before students must learn to think about sine and cosine as functions, varying with respect to an independent variable and having certain properties.

The right triangle approach to teaching trigonometry places a large emphasis on knowledge of procedures, specifically labeling triangles and computing ratios. Moreover, this procedural knowledge may come at the expense of conceptual understanding of what it means to

talk about sine and cosine as functions (Kendal & Stacey, 1997). Mnemonic devices such as SOHCAHTOA⁵ may actually have a detrimental effect on students' learning, because they put focus on memorizing a rule rather than making sense of a problem (Cavanagh, 2008). In addition, students who first encounter trigonometric functions as ratios in right triangles are more inclined to think that trigonometric functions take right triangles, rather than angles, as their inputs (Thompson, 2008). Students may struggle to translate from right triangle trigonometry to unit circle trigonometry, especially making connections between the triangle and circle representations (Bressoud, 2010; Thompson, 2008; Thompson, Carlson, & Silverman, 2007). Difficulties in this transition may stem back to what researchers have identified as misconceptions⁶ about angle measure, particularly in the connection between degrees and radians for measuring angles (Akkoc, 2008; Moore, 2013). The difficult transition from right triangle trigonometry to the unit circle is especially problematic given that the unit circle provides the basis for understanding trigonometric functions.

Recent research into students' learning of trigonometric functions has focused on identifying teaching activities to support students to depart from "memorization of isolated facts and procedures and paper-and-pencil tests and [towards] programs that emphasize conceptual understanding, multiple representations and connections, mathematical modeling, and problem solving" (Hirsch, Weinhold, & Nichols, 1991, p. 98). For example, Weber (2005) compared two different groups of students taking a college trigonometry course. One group had been taught within a traditional lecture-based course, where trigonometric functions were introduced in the

⁵ SOHCAHTOA is a common mnemonic device in American mathematics classrooms, used for remembering that the sine of an angle is found from the ratio of the opposite side over the hypotenuse, the cosine is the ratio of the adjacent side over the hypotenuse, and the tangent is the ratio of the opposite side over the adjacent side.

⁶ The cKc framework does not use the terminology of "misconception," based on the assumption that students use conceptions that work in certain situations.

typical progression from right triangles, to the unit circle, and then as functions. The experimental group learned about trigonometric functions first with a unit circle representation, and then going back and forth between Cartesian graphs and right triangles to culminate in graphing sine, cosine, and tangent functions. The students who participated in the experimental condition performed significantly better than students who received traditional instruction both on a post-test and in individual interviews. Students from the experimental group appeared to have better understanding of trigonometric functions, while students in the control group displayed very little understanding of trigonometric functions. Bressoud (2010) performed a similar study to Weber's, in which he introduced unit circle trigonometry before right triangle trigonometry. Bringing the initial focus away from the computations performed on right triangles seemed to support students in developing more conceptually-based understanding of trigonometric functions. Both studies suggested that students' conceptual understanding of trigonometric functions was fragile, even after having studied the concept. However, changing the ways in which students constructed and translated among representations had a positive impact on their learning.

There is some evidence that using technology may support students' learning about trigonometric functions. Blackett and Tall (1991) conducted an experiment in which two groups of students studied trigonometric functions under different conditions. The experimental group explored numerical and geometric relationships with computer software that drew right triangles with different conditions to facilitate students' explorations of the relationships between numerical and geometric quantities. The control group took a usual trigonometry course at the school. Students in the experimental group outperformed those in the control group on a post-test about trigonometric functions. This finding was promising in terms of using technology to

learn about trigonometric functions, although research in this area has been relatively sparse. Since technology tools often provide novel ways to construct and connect representations, one could expect that students who use technology in their study of trigonometric functions may gain stronger understanding of this concept.

Overall, it seems that there are two primary challenges for students studying sine and cosine functions. First, when students study sine and cosine ratios in right triangle trigonometry, the connections between geometric figures and numerical relationships create challenges for students in making connections between different representations. The most significant hurdle for students seems to be the transition from studying sine and cosine ratios on right triangles to studying sine and cosine functions on the unit circle and then on the coordinate plane. There is some evidence that working within a technological context, in which visual representations are readily available to students, may improve students' understanding of trigonometric functions (Blackett & Tall, 1991). As of yet, it is relatively unexplored whether programming activities that emphasize the connections between symbolic and graphical representations of trigonometric functions can support students' learning in this area.

Theoretical Framework

There are two distinct, but interrelated theories that inform this study. The first is the cK ϕ framework for *conceptions* (Balacheff & Gaudin, 2002, 2003). The second is the theory of *instrumented activity* (Verillón & Rabardel, 1995). The conceptions framework provides a way to identify students' conceptions of sine and cosine functions, while the theory of instrumented activity allows me to give more explicit attention to how students' use of Etoys contributed to those conceptions.

Conceptions Framework for Examining Students' Understanding

The theoretical framework for this study is the cK ϕ (pronounced “c-k-c”) model of conceptions (Balacheff & Gaudin, 2002, 2003, 2009). The cK ϕ acronym stands for “conception, knowing, concept,” each of whose meaning I will examine in turn. In mathematics, a *concept* refers to “the expression of a mathematical idea within the reasoned mathematical discourse of the discipline of mathematics, for example as part of a mathematical theory displayed in a mathematical text” (Herbst, 2005, p. 16). In other words, a concept is a mathematical idea that is widely accepted as such by the mathematics community. For example, a definition of a function as a rule, which takes some domain of inputs and maps each element of its domain to only one output, is a concept. The necessity of studying students' conceptions grows out of the fact that students' understandings of concepts in mathematics are not always aligned with the accepted mathematical thought.

Mathematics education researchers have examined students' conceptions to understand how students develop ideas about concepts in mathematics and why students' ideas are sometimes inconsistent with the accepted mathematical thought. As I mentioned in the introduction of this chapter, conceptions generally refer to students' beliefs, theories, meanings, and explanations of mathematics concepts (Confrey, 1990). In much of the work on students' conceptions, the definition of “conception” has been left implicit. Students' conceptions have been referred to alternatively as “naïve theory,” “private concepts,” “beliefs,” or “the mathematics of the child” (Confrey, 1990). The cK ϕ framework provides an explicit definition of the term conception. The framework transforms the construct of conception from a construct about mental activity to a collection of observable actions. In order to attend to the observable actions individuals perform that may reveal something about their thinking, Balacheff and

Gaudin (2003, p. 10) established the cK ϕ model for conceptions, where a *conception* is defined as a quadruplet $C = (P_C, R_C, L_C, \Sigma_C)$:

- P_C is the set of problems or tasks in which the conception is used,
- R_C is the set of operations that a student may use in completing the problems in that set,
- L_C is the system of representations in which the problem is posed and their solutions are expressed,
- Σ_C is the control structure, or the way of knowing whether the solutions expressed in the set of problems is correct.

The first three components of the cK ϕ framework came from Vergnaud's (1982, 1983, 1998, 2009) work on conceptual fields and his definition of concept. Vergnaud suggested that *problems* or *problem situations* were the source of students' understanding. Balacheff and Gaudin (2002) referred to a problem as a "perturbation of the subject-milieu system" (p. 2). In this definition, a subject is an individual, and the milieu is the subset of the environment relevant for a given piece of knowledge. The milieu is the antagonist system in the learning process (Brousseau, 1997). Practically speaking, problems, and specifically problems in mathematics, can be thought of as situations that calls for the use of mathematics. Traditional problems found in mathematics textbooks, as well as open-ended problems that students increasingly work on in mathematics classes, are examples of problems according to the definition of Balacheff and Gaudin. Different problems may call for different uses of sine and cosine functions, which would elicit different conceptions.

The operations of a conception are the "tools for action" (Balacheff & Gaudin, 2002, p. 7). Operations can be concrete or abstract. For example, operations could include the activity of programming a script with a particular syntax. More abstractly, an operation could be to apply a

specific theorem, or to transform an algebraic expression. When students work to solve a particular problem, operations are the things that students do, or the actions they perform, to solve the problem.

The system of representations acts as “the interface between the subject and the milieu” (Balacheff & Gaudin, 2002, p. 7). When students work on problems, representations are what allow students (the subjects) to interact with the mathematics environment. Representations include “diagrams, algebra, graphs, and tables” (Vergnaud, 1982, p. 36) as well as geometric drawings and spoken language (Balacheff & Gaudin, 2002). In addition, representations include the interfaces of technologies such as computer software or calculators. Representations are an integral component of a conception, because representations give account of the problem and allow the subject to perform operations. For example, algebraic manipulations are performed with symbolic representations of algebra. When a student performs an operation of graphing a function, the student produces a graphical representation that reflects the operation.

The control structure is the final component of a conception. A control structure provides the means for judging whether a solution provided to a particular problem is appropriate, from the perspective of the student. The idea of control structure in the cKc framework follows largely from work identifying the crucial role of control in problem solving (e.g., Schoenfeld, 1985). Schoenfeld argued that, when students solve problems that are not immediately obvious, they must have way of determining that their activities are appropriate for solving the problem. A control structure could include, for example, checking whether a solution matches a general formula, or checking whether a graph has the properties required in the posing of the problem. In school mathematics, students often have external control structures, for example asking a teacher or checking a solution in the back of a textbook. Control structures, particularly when

they do not come from an external source such as a teacher or textbook, are often left implicit in students' work. When students are encouraged to talk with one another, for example during pair work, students' control structures can become more explicit through their conversations with peers (Balacheff, 2013). A control structure is a crucial element of a student's conception, because it serves as the motivation for performing certain operations to solve a problem (Balacheff & Gaudin, 2002).

With the definition presented here, it is not surprising that students can have many different conceptions of a particular concept. Previous research with the cK ϵ framework has documented the different, and sometimes contradictory, conceptions that students display in the study of area (Herbst, 2005), congruence (González, G., & Herbst, 2009), solids of revolution (González, G., Eli, & DeJarnette, under review), and infinite series (Martínez-Planell, González, A. C., DiCristina, & Acevedo, 2012), as well as teachers' conceptions of tangent lines (Páez Murrillo, & Vivier, 2013). Students use different conceptions of symmetry during construction tasks than they do during proving tasks in geometry (Miyakawa, 2004). In addition, the problems provided in mathematics textbooks may elicit many different conceptions of a concept, for example in the case of functions (Mesa, 2004). When a student has multiple, and maybe many, different conceptions of a specific concept in mathematics, all of those conceptions contribute to the student's *knowing*. A knowing is "a set of conceptions which refer to the same content of reference" (Balacheff & Gaudin, 2002, p. 18). The use of "knowing" as a noun (following Brousseau, 1997) provides a distinction between an individual's personal constructs and what is deemed "knowledge", which refers to a construct recognized by a larger body such as the mathematics community (Balacheff & Gaudin, 2002; Herbst, 2005).

Drawing on Bourdieu (1990), Balacheff and Gaudin argued that individuals who encounter the same concept in different settings may treat that concept very differently. On one hand, an individual's actions are separated across time, so that contradictory conceptions of a particular concept do not have to face another. Consider, as an example, students' experiences with sine and cosine. In geometry, students may study sine and cosine as properties of angles; in right triangles, the sine of an angle is the ratio of the length of the side opposite the angle over the length of the hypotenuse. In this case, a student's conception of sine may be restricted to apply only to those angles that can occur in a triangle, and sine may be thought of as a property of an angle. At a later time in an Algebra 2 class, the same student's encounters with the concept of sine and cosine may be through studying sine and cosine functions. In this setting, sine and cosine can take all real numbers as their domain, which is contrary to students' prior knowledge of applying sine and cosine only to right triangles. Since these two conceptions of sine and cosine appear at different time periods in students' work, there may be no reason for students to confront their contrasting nature.

More importantly than just the time period, students' conceptions are tightly engrained within different domains of validity. A *domain of validity* of a conception refers to a collection of problems in which an individual's conception serves as an efficient way for thinking about a particular concept. Balacheff and Gaudin gave an example of this with the concept of decimal numbers: "Decimal numbers are not natural numbers with a dot, but to consider them as such is quite useful insofar as computation is concerned" (2002, p. 4). Returning to the earlier example, considering sine as a ratio is a practice largely engrained within the context of learning about sine and cosine in right triangles. Sine and cosine functions cannot be reduced to ratios, but it is useful to consider them that way when working with triangles. A conception of sine as a

function would not be useful if given a triangle and asked to compute the sine of an angle. Conversely, there is less use for a conception of sine as a ratio when one is studying sine as a function on the real numbers. The critical point here is that students can hold multiple conceptions of a single concept, and those conceptions may appear contradictory to an outside observer. However, as long as those conceptions are applied within different domains of validity, students need not face the contradictory nature of the conceptions.

There may be many domains of validity for working with sine and cosine functions. While these domains may be contradictory to one another, this does not create a problem for students if their work is situated in different domains at different times. Following the work of Bourdieu (1990), Balacheff and Gaudin used the phrase *spheres of practice* to designate mutually exclusive domains of validity. Different spheres of practice would require different conceptions from students. Moreover, although two spheres of practice may appear basically the same to an outside observer (e.g., a teacher), in the case of students' conceptions it is the student who determines whether particular conceptions will apply in a new sphere. A conception exists within a specific domain of validity, in which it is efficient for solving a problem. Transferring a conception from one setting to another is not necessarily an obvious process for students, even if it is obvious from the perspective of an observer (Balacheff & Gaudin, 2003, p. 3). This point is important, especially when considering students' prior knowledge as it applies to their work on a new problem. Although a new problem may appear to an observer as structurally the same as previous problems, students may not recognize it as such and may not apply their prior knowledge in a way that one would expect.

Instrumented Activity

The tools students use for learning mathematics—and particularly the technology tools they use—shape the ways that students think about mathematics (Bartolini Bussi & Mariotti, 1999; Meira, 1995; Noss & Hoyles, 1996; Pea, 1987). This understanding is based on a more general idea that individuals' uses of tools change the nature of all human activity (Verillón & Rabardel, 1995). A *tool* in this sense is any physical artifact that has been designed and agreed upon to serve some specific purpose (Leontiev, 1981). While an artifact could be any physical material to which an individual has access, a tool is meant to be used in a specific way.

It is critical to examine the different ways students use technology tools for mathematics, because the ways that students use tools impact how they think about mathematics. A clear example of this comes from the case of dynamic geometry environments, and students' use of the *dragging* tool (Arzarello et al., 2002; Hollebrands, 2007; Hoyles & Noss, 1994). The dragging tool in a dynamic geometry environment is a feature of the environment that allows students to drag already-constructed geometric figures. Dragging allows the figure to maintain the properties of the figure, although its size or orientation might change. For example, a student could construct a square and then drag a vertex of the square to dilate, rotate, or translate it. Dragging the square does not change its defining properties, namely four right angles and four equal sides.

The different ways that students might use the dragging tool in a dynamic geometry environment can result in differences in how students reason about geometric figures. Hollebrands (2007) found that students taking a geometry course with a dynamic geometry environment used the dragging feature to test their geometric constructions, to test their conjectures, and to examine invariants of a figure. The ways that students interpreted the results

of their dragging varied, from using the drag test to determine the critical properties of a sketch, to using a drag test simply to describe a sketch. In some cases, students' use of the dragging tool allowed them to identify the defining features of a geometric construction. In other cases, dragging served a less integral role, just giving another way to talk about a sketch. This finding is enlightening, because it suggests that even a tool designed for a specific purpose is likely to be used and interpreted differently by different people.

As of yet, little research has examined specifically what tools computer-programming environments provide for learning mathematics. Research on students' use of programming environments has focused largely on the connections among representations afforded by these environments, and the language offered by the programming syntax for talking about mathematics (e.g., Clements & Battista, 1989, 1990; Edwards, 1991, 1997; Hoyles & Healy, 1997; Hoyles & Noss, 1992). One could expect that these advantages provided by programming environments may be the result of specific tools that are available in those environments. For example, many student-friendly computer-programming environments include a *drag-and-drop* feature, which allows users to drag tiles and put them together to construct a program. This drag-and-drop feature is a tool, inasmuch as it is designed for the specific purpose of supporting novice programmers to overcome difficulties with syntax and construct workable programs. Similarly to the work that has been done in dynamic geometry, identifying specific tools available in a programming environment can offer some insight into how students' use of those tools supports their learning about mathematics.

Instrumented activity refers to the activity that occurs when an individual appropriates a tool or artifact and integrates it into his or her activity (Verillón & Rabardel, 1995). The concept of tool appropriation comes largely from the work of Leontiev (1981) and Vygotsky (1978). The

appropriation of a tool requires the user of that tool to reproduce the activity “adequate to the human purpose it embodies” (Leontiev, 1981, p. 263). This means that to appropriate a technology tool for learning mathematics, a student must integrate that tool into his or her activity in a way that is recognized as a practice of doing mathematics. Any student could use, in a literal sense, the drag-and-drop tools of a programming environment, picking up tiles and moving them around the computer screen. However, as Vygotsky (1930) pointed out, as an individual appropriates a tool, he or she must engage with the social nature of that instrument, the way it has been designed for some use. Tool appropriation is not just a physical activity, or a cognitive activity, but it is a social activity. When students appropriate technology tools in mathematics, they participate in, and contribute to, a socially determined purpose for the use of those tools.

A tool becomes an *instrument* when a person has appropriated it into his or her activity in a way that has meaning for achieving a specific purpose. Instrumented activity is not a phenomenon that is unique to students’ learning in mathematics. Verillón and Rabardel (1995) provided the example of a young child learning to use a spoon to eat as an example of instrumented activity. A spoon is a tool designed very specifically, for a specific use. However, the spoon does not become an *instrument* for the child until that child uses the spoon to achieve a specific goal, namely eating. While an artifact or tool is any physical construct, an instrument has meaning attached to it. An instrument includes not only the physical object, but also a person’s understanding of how and why to use that object.

Instrumented activity is especially important to consider in the case of students’ learning of mathematics, because the different ways that students appropriate the same tools have implications for their learning. In the example of the dragging tool from Hollebrands (2007),

different students appropriated the dragging tool for quite different purposes. In some cases, students used dragging only to describe the constructions they had made in the dynamic geometry environment. In other cases, students used the dragging tool for a more sophisticated purpose, to determine the critical features of a construction to create a given figure. Both uses of the tool were an acceptable use of how the dragging tool was defined. But the different ways that students appropriated the tool gave them different ways to examine and understand the mathematics at hand.

In the case of programming environments, the different ways students appropriate the tools available could similarly impact the nature of students' thinking about mathematics. To explain the dynamics of instrumented activity, Verillón and Rabardel (1995) proposed the Instrumented Activity Situations [IAS] model (see Figure 2.2). In the IAS model there are three key components: a subject (e.g., a student), an object of the subject's activity (e.g., some mathematical concept), and the instrument (e.g., a tool provided in a programming environment). Instrumented Activity is comprised of a collection of pairwise interactions. Specifically, the subject interacts with the tool and also directly with the object. In addition the tool itself interacts with the object. Finally, the subject interacts with the object in a way that is shaped by the tool.

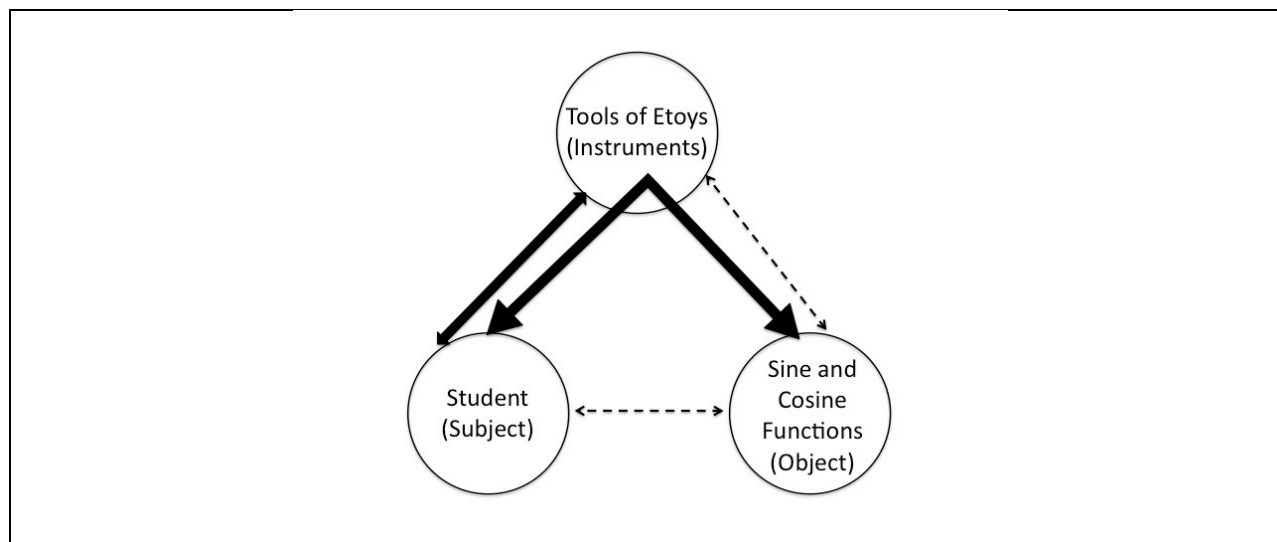


Figure 2.2. The Instrumented Activity Situations [IAS] model (Verillón & Rabardel, 1995). The dotted lines indicate interactions that are present in the model but are less integral to this study. The bold black lines indicate interactions that are integral to this study.

For the purpose of this study, the critical interactions in the IAS model are the interactions between students and Etoys (subject-instrument) and the interactions between students and sine and cosine functions *through* their use of Etoys (subject-instrument-object). While focusing on these interactions, I recognize that the other aspects of the model were present in students' work. Students in this study had prior knowledge of sine and cosine functions apart from Etoys or any other technology tools. The interactions between the tools and the mathematical concept are less obvious, but were certainly still present in students' activity. For example, Etoys was designed with a definition of sine and cosine functions already programmed into the software. This feature made drag-and-drop tiles available for sine and, respectively, cosine, which gave students opportunities to program with those functions. For this study, the subject-instrument and subject-instrument-object aspects of the IAS model were the most integral to examining how students built understanding of sine and cosine functions through their use of Etoys.

Data and Methods

The data for this study come from the 2-day Etoys lesson that Ms. Alexander⁷ taught in her three sections of Algebra 2. Ms. Alexander is a teacher at Grove High School, a school of approximately 1000 students, with around 60% White and 30% Latino/a students. Thirty percent of students at Grove High School qualify for free or reduced lunch. The students in Ms. Alexander's classes were not on the honors track at the school. Most of the students in the study were in 11th grade, and a small number of students were in 12th grade. All of the students in all three sections participated in the Etoys lesson, although not all students participated in the study.

Ten pairs of students and one group of three, spread across the three class periods, participated in the study (Table 2.1). The data for analysis come from the video and audio transcripts from students' work in pairs over the two days of the Etoys lesson, as well as copies of students' written work and records of students' work on the computer during the two days of the lesson. For each pair of students, I positioned an audio recorder between the students. I also positioned a video recorder angled to record the students and the computer screen. The most important aspect of the video recording was to capture who was talking or writing at each moment, and also when students traded places at the computer or pointed to things on the computer screen. In my analysis, I focused on the mathematical ideas that were conveyed through students' talk and written work.

⁷ I use pseudonyms throughout.

Table 2.1
Students Participating in the Study of the Etoys Lesson

Carson Abbey	Jalisa Zach	Gia Courtney	Hannah Dayana	Shane Maya	Lucas Elizabeth Andy
Bailey Aubrey	Cara Maggie	Mike Jessa	Mitchell Reese	Tori Sean	

The Ferris Wheel Problem

The problem that students worked on during the Etoys lesson was the Ferris wheel problem. Throughout this study, I will refer to the “Ferris wheel problem” to refer to the problem itself and students’ work on the problem. I will refer to the “Etoys lesson” to refer more generally to the teacher’s and students’ activities over the two days. Ms. Alexander taught the Etoys lesson towards the end of the spring semester. The purpose of the lesson was for students to use sine and cosine functions to represent real-world phenomena and to answer questions about real-world phenomena based on their representations. Earlier in the semester, students in Ms. Alexander’s class had studied sine and cosine functions, discussing aspects of domain and range, periodicity, amplitude, and shift. Students had graphed sine and cosine functions. However, they had not used sine and cosine functions to represent any real world situations. The Ferris wheel lesson was designed to target the standard that students would learn to “model periodic phenomena with trigonometric functions” (NGAC, 2010, p. 71). Ms. Alexander welcomed the Etoys lesson as an opportunity to improve students’ abilities to visualize and create representations at the conclusion of their unit on trigonometric functions.

During the Etoys lesson students worked on the Ferris wheel problem (see Figure 2.3). On the Ferris wheel problem, students were to imagine that they were riding the London Eye in London, England. Students had to write a function to represent their height off the ground at

various times while riding the London Eye. The Ferris wheel problem was really comprised of two sub-problems. First, students had to plot points on a Cartesian plane to represent their height off the ground at various moments during their ride on the Ferris wheel. This first problem required students to translate from a visualization of the actual Ferris wheel, to a mathematical representation in the Cartesian plane, where the x -axis represented time and the y -axis represented height. Second, students had to construct a function that would overlap the points they had plotted in the first problem. To construct such a function required students to recognize that the points they had plotted could be represented by a sinusoidal graph. Then, students needed to use information about the amplitude, vertical shift, and period of the graph to determine the appropriate parameters for their function. See Appendix A for a copy of students' worksheet containing the Ferris wheel problem.

One of the most famous Ferris wheels in the world is the London Eye in London, England. Assume that the London Eye has a diameter of 130 meters, and the lowest point on the Ferris wheel is 5 meters above the Thames River. It takes 30 minutes to make one complete revolution. You and your partner are going to ride the Ferris wheel. You get on the Ferris wheel at the very lowest point.

Figure 2.3. The Ferris wheel context.

Since any sine function can be represented as a cosine function, and vice versa, students could have used either a sine function or a cosine function to represent their height off the ground while riding the Ferris wheel. Functions that would have worked to model the scenario in Figure 2.2 include $f(x) = -65 \cos\left(\frac{2\pi}{30}x\right) + 70$, $f(x) = 65 \sin\left(\frac{2\pi}{30}x - \frac{\pi}{2}\right) + 70$, or $f(x) = 65 \sin\left(\frac{2\pi}{30}x + \frac{3\pi}{2}\right) + 70$. The coefficient of 65 in front of each function accounts for the amplitude, or vertical stretch, of the graph. The coefficient of x inside each function accounts for the period of one revolution of the Ferris wheel, and the 70 added to end of each function

accounts for the vertical shift of the graph. For each of the sine functions, students would need to include a horizontal shift to model their height appropriately, which could be done either by subtracting $\frac{\pi}{2}$ or by adding $\frac{3\pi}{2}$ ⁸. Since the cosine function did not require a horizontal shift, I expected that most students would use a cosine function to represent the scenario.

Finally, to account for the fact that students in a computer lab would likely be seated very near to one another, I created two versions of the problem provided in Figure 2.2. In the second version, the Ferris wheel was described as 140 meters in diameter, and it took 40 minutes to make one complete revolution. An appropriate function to model this alternate scenario would be $f(x) = -70\cos\left(\frac{2\pi}{40}x\right) + 75$ or $f(x) = 70\sin\left(\frac{2\pi}{40}x - \frac{\pi}{2}\right) + 75$. Based on my request, when Ms. Alexander distributed the Ferris wheel problem in class, she gave alternating groups alternating versions of the problem. The purpose of this was to identify whether students' solutions or ideas came from listening to the conversations of the groups around them. I expected that, by providing groups next to each other different versions of the problem, I would have a cue about when groups were sharing ideas with each other.

Features of the Etoys Notebook

For their work on the Ferris wheel problem, students were encouraged to use Etoys. Students were provided with a pre-constructed virtual notebook in Etoys that was set-up with different pages for students to use during their work on the problem. I designed the Etoys notebook in consultation of individuals with expertise in implementing Etoys activities in educational settings (González & Lundsgaard, personal communication, 2013; Pitt, personal communication, 2013). Students could click through the virtual pages of the Etoys notebook in

⁸ In addition, students could have added any integer multiple of 2π to account for the shift, since sine and cosine functions are periodic.

whatever order they chose. The first page of the notebook included a virtual Ferris wheel (Figure 2.4). The virtual Ferris wheel was intended to help students visualize how high off the ground they would be at different moments in time. The virtual Ferris wheel included a label of “5 meters” off the ground at the bottom of the Ferris wheel. The notebook did not include any other numerical labels, and it did not account for the duration of time in any way. It was left to students to make those connections based on the information given in the problem.

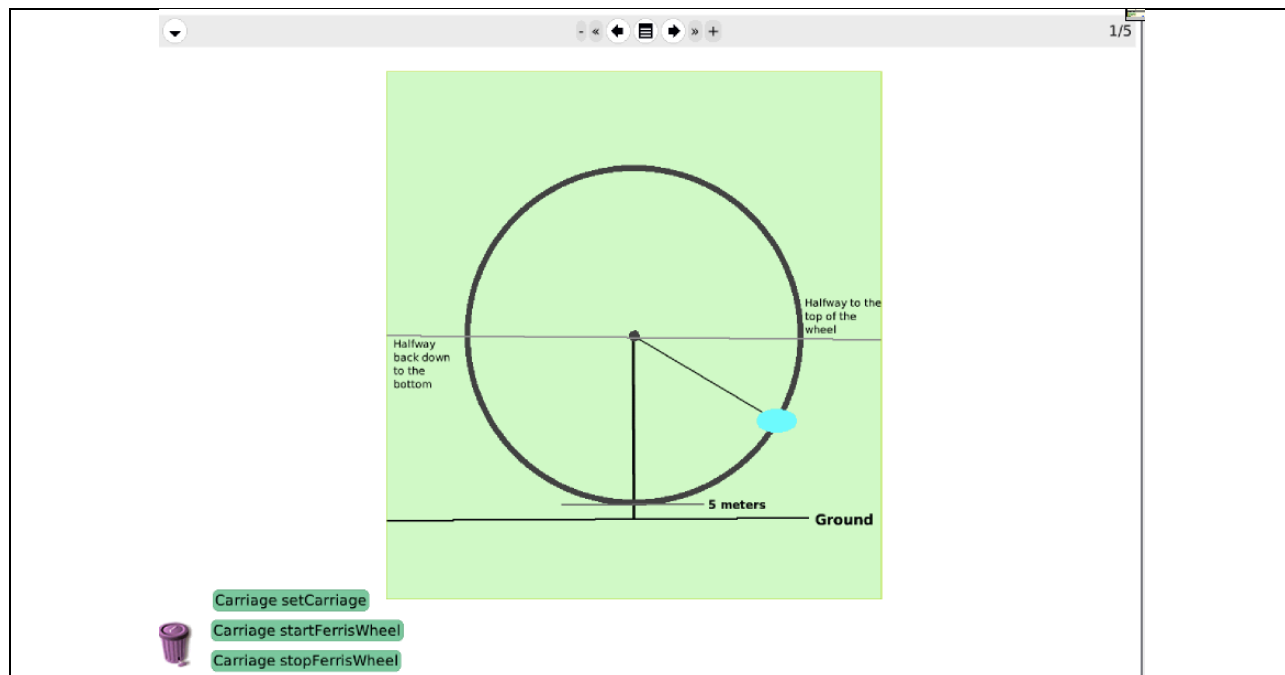


Figure 2.4. The virtual Ferris wheel representation on page 1 of the Etoys notebook.

The second page of the Etoys notebook included an example of a script that would plot a graph in Etoys (Figure 2.5). The purpose of this page was for students to have an example of a script in Etoys that would serve as a reference when students needed to write their own scripts. The page provided a green dot that was labeled the “plotter,” which served as a programmable object. There were three basic steps in the script. The first command in the script assigned a value to the y -coordinate of the plotter based on its current x -coordinate. In the example provided, the y -value assigned was equal to $2(x + 1)^2 - 50$. The second command of the script

told the plotter to make a “stamp,” which left a mark on the screen to indicate the plotter’s location. The final command of the script told the plotter to increase its x -value by 5, before running the script again. Students could experiment with this script, changing the values of the numbers and changing the function, as they needed to when it came time for them to construct their own functions.

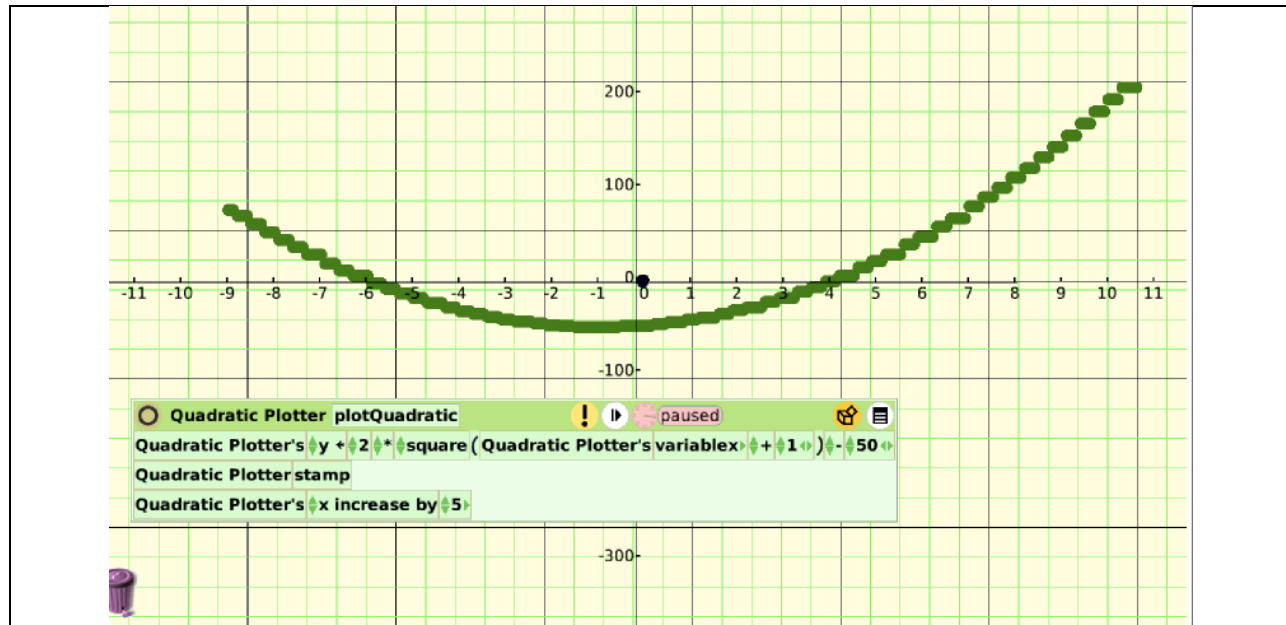


Figure 2.5. The quadratic script on page 2 of the Etoys notebook.

During the launch of the lesson on Day 1, Ms. Alexander spent approximately three minutes in each class period looking at page 2 of the Etoys notebook with students in the class. The purpose of this time was to introduce the student in the class to Etoys and to show them some essential features to get started on their work. First, Ms. Alexander showed students how they could change the values of numbers in the script, either by scrolling with the arrows or by clicking directly on a number and typing in a new value. Second, Ms. Alexander showed students that if they clicked on the tile labeled “square,” they could select from a variety of other functions, including absolute value, exponential, and logarithmic. Third, Ms. Alexander showed

students how they could drag other tiles into the script to replace the things that were there. Ms. Alexander wanted to point out to students that, in order to drag a tile, they needed to place it so that the area behind the tile turned bright green. That was the cue that students could use the drag-and-drop tiles to put together pieces of their script.

The final thing that Ms. Alexander did during the launch of the lesson was to show students how the command “Quadratic plotter’s $y \leftarrow 2 * \text{square}(\text{Quadratic plotter’s variable } x + 1) - 50$ ” would translate to standard algebraic notation. Ms. Alexander pointed out that “Quadratic plotter’s y ” represented the y -variable, and similarly for “Quadratic plotter’s x .” In addition Ms. Alexander noted for students that “ $\text{square}(\text{Quadratic plotter’s variable } x + 1)$ ” was equivalent to writing $(x + 1)^2$. Ms. Alexander wanted to point out for students that the meaning of the squaring function was the same, even though the function was represented somewhat differently. After Ms. Alexander’s launch, she encouraged students to experiment with page 2 of the Etoys notebook to examine whether they could change numbers or make different graphs. In the design of the lesson, I expected that students would refer back to page 2 at moments when they needed a reference for how to construct a script to represent their height while riding the Ferris wheel.

The third page of the Etoys notebook contained a Cartesian graph and some resources for students to construct a graph in Etoys (Figure 2.6). Students were provided with a set of blue points that they were to use for the problem of plotting discrete points to represent their height off the ground at various moments. They needed to plot these discrete points to determine the shape of the graph they would need to make. In addition, students were provided with a pink “plotter” and a collection of tiles that they could use to build their scripts. The collection of tiles provided to students included everything they would need to construct a script to plot a sine or

cosine function, plus several additional tiles. The purpose of providing tiles to students was so that they did not have to spend extra time navigating through all of the features of Etoys, many of which were not directly relevant to the lesson. I provided extra tiles beyond what students needed in order to not guide students to the appropriate solution to the problem.

To build the script on page 3 of the notebook, students would need to recognize that the blue points they had plotted could be represented by a sine or cosine function. Students would need to write a script, using a sine or cosine to assign a value to the y -coordinate of the plotter in the first command. They would need to use information about the height, width, and placement of the graph to determine the appropriate parameters for the function in their script. Ms. Alexander encouraged students to use Etoys to work on the problem, but students were not restricted to only using Etoys. Ms. Alexander distributed the Ferris wheel problem on a worksheet, which included space to write. She told students that they could do work with a paper and pencil as well, particularly if they thought they knew the solution to the problem but could not enter the correct function into Etoys. Ms. Alexander told students that, if they needed to, they could write their solution on paper, and she would help them enter the solution into their scripts.

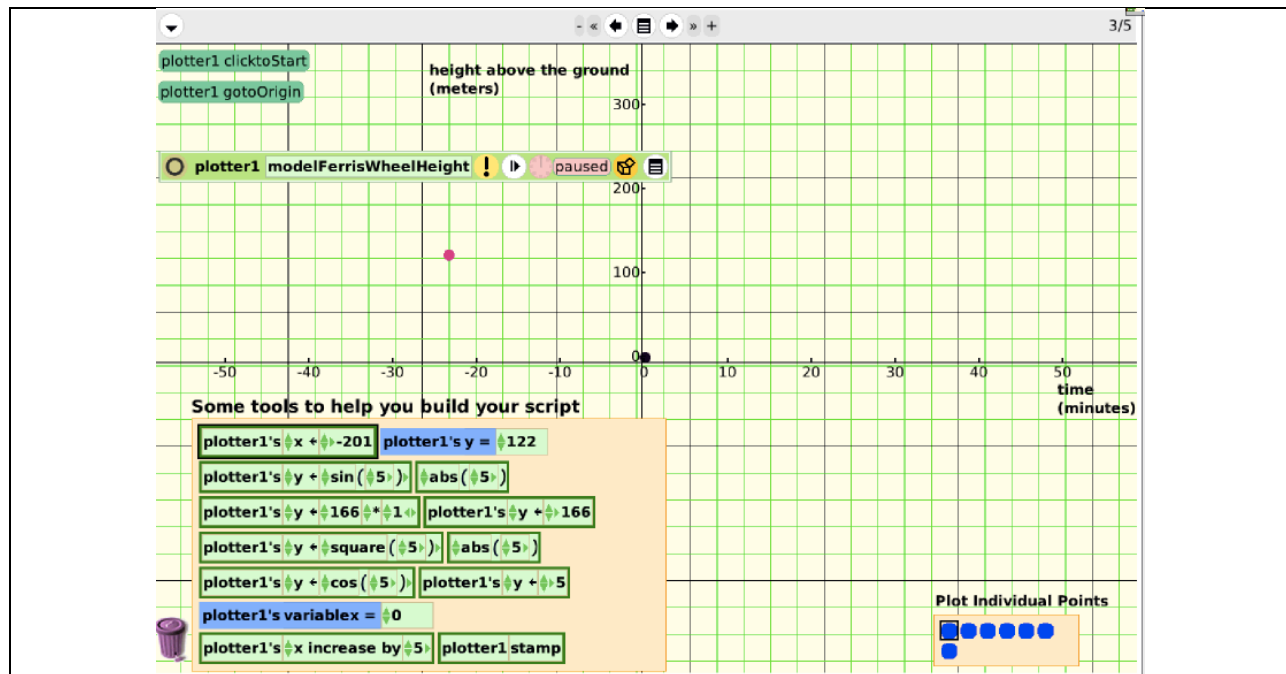


Figure 2.6. The features provided to students on page 3 of the Etoys notebook.

My role during the Etoys lesson was a participant-observer. When planning the lesson, I expected that I would act only as an observer, and I would serve as a resource to Ms. Alexander if she had questions about Etoys. When introducing me to her classes, Ms. Alexander told her students that they could ask me questions. While I was primarily an observer, there were times during the two days when pairs of students asked questions directly to me, either about the mathematics or about Etoys. When students had technical difficulties with Etoys unrelated to the mathematics at hand, I answered them directly. For example, students occasionally “lost” their plotters, because they disappeared off the screen. In instances such as that, I helped students resolve those issues so that they could get back to work on the problem. When students asked me questions about mathematics, I played the role of a facilitator, asking questions and clarifying students’ ideas, but not providing answers to questions. Students asked me relatively few questions. I primarily observed students in the study as they worked, and I took field notes of my observations.

The data offered two primary advantages for this study. First, students' work in this study was situated in their typical classroom context. This provided for consistency in prior knowledge across the students in the study and in students' understanding of the subject matter. Compared with if students had participated in an after school activity or an enrichment program, all of the students in this study had the same teaching for the entire school year prior to this lesson. Moreover, since this lesson was integrated as part of the usual unit on sine and cosine, students were held to the same expectations in terms of their class participation and their completion of the assignment as they would be if they were not participating in a research project. These conditions created a level of authenticity with regards to how well these data represent what types of learning may actually occur in a typical classroom.

Applying the Conceptions Framework

For each pair of students, I created a timeline of their work on the problem according to the different components of the conceptions framework (as in González, G. et al., under review). Table 2.2 provides a template for this timeline. Each of the four columns in the timeline indicates one of the components of the conceptions framework. Each row in the timeline indicates an instance of a conception. The first step in my analysis was to segment students' transcripts according the problems that they were working on in order throughout the lesson. When identifying the problem, it was important to identify the problem *as students perceived it*, which may or may not have been the same as the problem as it was designed. This distinction will become more evident with a forthcoming example. After identifying the problems that pairs worked on during the lesson, I gave a more fine-grained analysis of the operations, representations, and control structures present in students' work. I created a new row in the timeline, indicating a change in conception, each time that any of the components of the

conceptions framework changed. For example, it was possible that students persisted working on a single problem, but the operations they used for working on that problem changed. Since a change in any of the four components created a new quadruplet, I identified each change in any component of the conceptions framework as a distinct conception in the timeline.

Table 2.2

A Template for Coding Students' Conceptions During Work on the Ferris Wheel Problem

What is the <i>problem</i> that the students are working on?	What <i>operations</i> are the students using?	What <i>representations</i> are the students using in the <i>solution to the problem</i> ?	What <i>control structure</i> do the students use in order to know that the solution is correct?
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Example of the Analysis

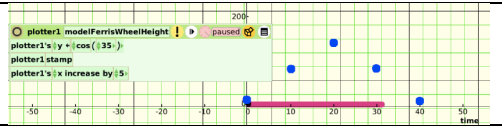
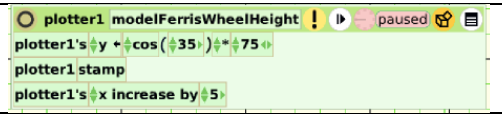
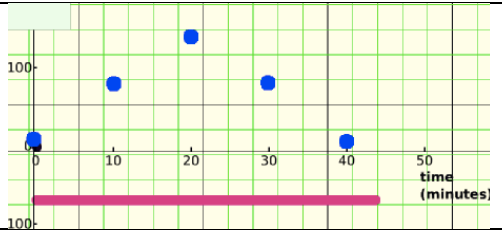
The transcript in Table 2.3 provides the basis for an example of the analysis according to the cK ϕ framework. The transcription conventions in Table 2.3 will apply to all the transcripts in the chapter. The leftmost column of the transcript refers to the turn number within the segment. I will use the turn numbers to refer back to sections of the transcript in my analysis. The second column from the left indicates the speaker. Throughout, I will use Ms. A to refer to Ms. Alexander in transcripts. I will use AD to refer to myself in transcripts. In the column for the speaker's turn, comments in brackets indicate the actions of a speaker or to whom their talk is directed. Ellipses at the end of a statement indicate that the speaker trailed off without completing a statement or question. The symbol “-“ indicates that a speaker interrupted his or her own statement with a new statement, without pausing. In cases where a speaker paused in the middle of a turn, I use brackets to indicate the length of a pause. For instance, [pause 3 sec] would indicate a pause of three seconds in between words. Pieces of the transcript that are bolded are statements that specifically revealed one or more components of the conceptions

framework. Bolded text in the transcripts does not indicate any emphasis on the part of the speaker, but rather it indicates part of the statement that was particularly relevant to me in my analysis. The rightmost column of the transcript indicates the work that was present on the students' computer screen at different moments in the transcript. For columns that do not contain pictures of the screen work, the reader can assume that the screen work has remained the same since the previous picture. See Appendix H for a summary of the transcription conventions.

The transcript in Table 2.3 comes from the conversation between Tori and Sean on Day 1 of the Etoys lesson. Tori and Sean were working on the version of the Ferris wheel problem in which the diameter of the Ferris wheel was 140 meters and the time to make one revolution was 40 minutes. In the transcript below, Tori and Sean had already plotted their points, and they had decided that they would need either a sine function or a cosine function to represent their height off the ground.

Table 2.3

A Discussion Between Tori and Sean During Day 1 of the Etoys lesson

Turn	Speaker	Turn	Screen Work
1	Tori	Okay, where does this 140 come in at? Does that come in anywhere?	
2	Sean	I don't know. I don't think so.	
3	Tori	Do we use 70 somewhere? 75?	
4	Sean	[To Ms. Alexander] How do we put a number in front of this? [Pointing to the cosine function in the script]	
5	Ms. A	Okay, so you can't put a number in front, but you can put it behind. Like, what are you trying to put?	
6	Sean	I don't know. Just a number.	
7	Tori	Seventy five.	
8	Ms. A	So, to put it in front, if you change that to - Cuz what do we do if we have, like, 2 cosine? We are...	
9	Tori	Multiplying.	
10	Ms. A	Multiplying. So change that to multiplication. And then you can put your number. And that'll be your number in front.	
11	Sean	Get rid of that.	
12	Tori	Is it 75? I think it's 75.	
13	Sean	[Enters 75.]	
14	Tori	Ah! Play. Moment of truth.	
15	Sean	[Runs script.]	
16	Tori	Oh my god.	
17	Sean	Why did that move down? 'Cuz shouldn't it - 'cuz, it multiplied, right? So it shouldn't, it shouldn't move that down.	

The segment began by Tori asking how they should use the information about the diameter of the Ferris wheel being 140 meters (turns 1-3). In turn 4 of the transcript, Sean began to reveal a conception of sine and cosine when he asked Ms. Alexander, “How do you put a number in front of this?” Sean’s question was related to a struggle he was having with the Etoys syntax. Specifically, he had used the drag-and-drop tiles to insert a cosine function into the first command of his syntax. At that point, Sean wanted to put a number in front of the cosine function. Sean’s question revealed the operation he was trying to use (put a number in front of the cosine function), and it also suggested something about the problem he was trying to solve. Specifically, it seemed that Sean was trying to construct a symbolic representation of a cosine function that was consistent with how he had written cosine functions in the past. This was important for understanding the problem that Sean was trying to solve. In the design of the lesson, the problem was to construct an appropriate function that would have the necessary characteristics to run through the discrete points. For Sean, the problem was to construct a representation of a cosine function that matched the symbolic representations he had used in the past. Students in Ms. Alexander’s class had become accustomed to writing cosine functions as $y = a\cos(bx) + c$. Sean was attempting to construct a function like this in Etoys, although the syntax prevented him from doing so.

Sean’s conception during his conversation with Tori and Ms. Alexander was made more explicit when Ms. Alexander asked Sean what number he wanted to put in front of the cosine function (turn 5). In response, Sean said he did not care what number, he just wanted to put any number in front (turn 6). Although Tori indicated some attempt to use the information about the diameter of the Ferris wheel, Sean did not make any connection between the symbolic representation and the other representations (plotted points, graph, or otherwise) of the problem.

With the conception present in turns 1-7 of the transcript, the problem that Sean and Tori were trying to solve was to write a function that looked like a typical cosine function. To do so, they wanted to put a number in front of the cosine function they had already chosen. They relied on the verbal description of the problem, their prior knowledge of the written-symbolic algebraic notation, and the drag-and-drop tiles that were provided in Etoys. The control structure of this conception was that cosine functions should follow a certain written representation. Using the Etoys syntax, Sean could not satisfy the control structure of the conception. This control structure was what had provoked Sean to ask Ms. Alexander a question about putting a number in front of the cosine function.

During the conversation in Table 2.3, Sean and Tori changed their conception of sine and cosine, reflected by a change in all the components of the conceptions framework. This shift was provoked by three different factors. First, Tori had been arguing since the beginning of the segment that they should account for the diameter of the Ferris wheel (turns 1, 3, 12). Although its not clear why Tori decided to multiply by 75 rather than 70, it could have been because the highest point the Ferris wheel reached was 145 meters off the ground. Regardless, Tori indicated in turns 1 and 3 that she was taking into account the diameter of the Ferris wheel to determine the coefficient of the cosine function. Ms. Alexander's question about the operation between a number and the cosine function (turns 8-10), along with Sean's inability to drop a tile in front of the function, seemed to move Sean past his initial conception.

The problem to be solved changed from writing a function that looked like a typical cosine function, to determining how to vertically stretch the cosine function to reach the highest and lowest of the plotted points. Again, this shift in the problem was evidenced by Tori's insistence on paying attention to the diameter of the Ferris wheel for determining the coefficient

of the function. The operation, rather than putting a number in front of the cosine function, was to multiply the cosine function by 75. Also, with this new conception, the representations included the output of the Etoys script (turns 13-16). After performing this new operation, with the new representation, Tori and Sean had a new measure of control for determining the correctness of their solution. The graphical output of the Etoys script allowed Tori and Sean to test whether their graph overlapped the discrete points they had plotted.

The analysis that I have described is summarized in Table 2.4. The first row of Table 2.4 indicates the first conception that I identified in the segment between Tori, Sean, and Ms. Alexander. The conception began in turn 1 and progressed through turn 7. The second conception overlapped with the first conception, beginning in turn 3 when Tori made an explicit request to use the information about the diameter of the Ferris wheel. The operations, representations, and control structure of each of the two conceptions followed from each of the problems. The numbers in parentheses in Table 2.4 indicate the turns of the transcript that gave evidence of each component.

Table 2.4

An Analysis of Tori and Sean's Conversation According to the cK ϕ Framework

Turns	What is the <i>problem</i> that the students are working on?	What <i>operations</i> are the students using?	What <i>representations</i> are the students using in the <i>solution to the problem</i> ?	What <i>control structure</i> do the students use in order to know that the solution is correct?
1-7	Write a function that looks like a typical cosine function (e.g., $y = \text{acos}(bx) + c$).	Divide the height of the Ferris wheel by 2 (1, 3) Put a number "in front" of the cosine function (2, 4-7)	Verbal description (1) Written symbolic algebra Etoys drag-and-drop tiles (4-5)	Etoys syntax would not allow students to "put a number in front" (4-5)
3-17	Given a sinusoidal function and a set of discrete points, stretch the graph vertically so that the graph reaches the maximum and minimum points.	Divide the height of the Ferris wheel by 2 (3) Multiply the cosine function by 75 (3, 12)	Verbal description Plotted points Etoys drag-and-drop tiles Output of Etoys script	Multiplication should not moves things down (17)

It is important to note that I considered conceptions of sine and cosine functions as a characteristic of a pair or group of students, rather than as a characteristic of an individual student. In some cases, as in the case above, this meant that conceptions were sometimes overlapping within a pair or group. Also in some cases, as in the case of Tori and Sean above, it was more evident that a conception was coming more from one student than the other. However, since students were working in pairs, and talking to each other, it would not have been feasible to entirely disentangle individuals' conceptions and identify conceptions as belonging solely to one individual or another. For this reason, I identified a conception as surfacing in a pair if the conception became apparent in students' talk or actions on the computer. Since this is a study of

students' understanding of sine and cosine through their work in pairs, it was appropriate to keep the unit of analysis at the group level, and not at the individual level. Allowing for conceptions to overlap, as in Table 2.4, allowed for the consideration that students may have needed to use multiple conceptions to find the solution to the problem. Also, I accounted for the possibility that multiple conceptions could surface in students' talk at the same time.

Finishing the Conceptions Analysis

After coding all of the segments, I aggregated similar segments within pairs in order to identify a smaller, more meaningful collection of conceptions that students used throughout their work on the lesson. The most critical components that I examined to compare different conceptions were the operations that students performed, and the representations they used to perform those operations. Aggregating the similar segments according to the commonalities among these components allowed me to identify characteristic features of the different conceptions that students used throughout their work. For example, I identified several instances where students attempted to reproduce a standard written representation of sine and cosine functions, as was the case in the example in Table 2.3. Aggregating conceptions according to commonalities yielded distinct conceptions of sine and cosine that surface during students' work on the Ferris wheel problem, which I labeled according to their characteristic features.

The final step in the analysis was to check my coding for reliability. To determine reliability I worked with another graduate student who familiarized himself with one randomly selected transcript from Day 1 of the Etoys lesson and one randomly selected transcript from Day 2 of the Etoys lesson. Each of the transcripts represented a 5-minute segment of students' work. Before providing the other coder with the coding scheme, I gave him time to read through the transcripts, watch the videos, and ask clarifying questions about students' conversations or

actions. Since the other coder had not been present during the Etoys lesson, and was not himself an expert with Etoys, I wanted to provide as much opportunity as possible to clarify the content of the transcripts before checking my coding for reliability. Initially, I provided the other coder with a list of the seven conceptions I had identified, along with a brief (2-3 sentences) description of each conception. The coder used the list to analyze the Day 1 transcript, and I compared our coding. Of the conceptions that I had identified in the transcript, we agreed on 60% of those conceptions after the initial check for reliability.

After the first round of reliability checking, I suspected that the discrepancies in our coding resulted because the other coder did not have enough information about each of the conceptions. To resolve this, I provided the other coder with the list of each of the conceptions, along with the problems, operations, representations, and control structures of each conception (as in Mesa, 2004). The coder used this outline to analyze the Day 1 transcript and we reached a reliability of 82%. In addition, the other coder used the outline of the conceptions to analyze the Day 2 transcript, and we reached a reliability of 84%. Following other checks of inter-rater reliability using the conceptions framework (e.g., Mesa, 2004), I deemed this level of reliability appropriate.

The conceptions framework allowed me to consider how students engaged with the topic of sine and cosine functions through their observable actions. In applying the conceptions framework I was able to analyze what students did in response to what the problem called for. In that way, I was able to understand how students may have invoked different, and even conflicting, conceptions of sine and cosine in ways that did not appear contradictory to students at the time. With the cKc framework I was able to better understand how what otherwise may be

deemed “misconceptions” may actually have viable strategies for students based on their perceptions of the problem.

Results

In this section I outline the results of the analysis according to each of the three research questions guiding this study.

Students’ Conceptions of Sine and Cosine

By performing the conceptions analysis, I sought to answer my first research question: *What conceptions of sine and cosine did students invoke throughout their work on an open-ended problem with Etoys?* I identified seven different conceptions of sine and cosine functions in students’ work, which I organized according to the viability of those conceptions for solving the Ferris wheel problem. Viability is a construct from constructivist theories of learning. Determining whether students’ conceptions are “right” or “wrong” is not helpful when working from the assumption that all conceptions have some domain of validity. What is more important is whether, from the perspective of the learner, particular conceptions are *viable*, or in other words whether they work for solving a problem (von Glasersfeld, 1993). In my analysis of students’ conceptions, I identified a hierarchy of conceptions based on which conceptions were more viable for students to solve the two sub-problems of the Ferris wheel problem.

Figure 2.7 gives an overview of the seven different conceptions I identified, organized according to the two sub-problems that composed the Ferris wheel problem. For the first problem, where students had to use the representation of the Ferris wheel to plot points representing their height of the ground, I identified two conceptions. Those two conceptions were distinct in terms of their viability for solving the problem of plotting the points. On the second problem, where students needed to construct a graph to run through the points they had

plotted, I identified five distinct conceptions of sine and cosine. Those five distinct conceptions fell into three different levels of viability for solving the Ferris wheel problem.

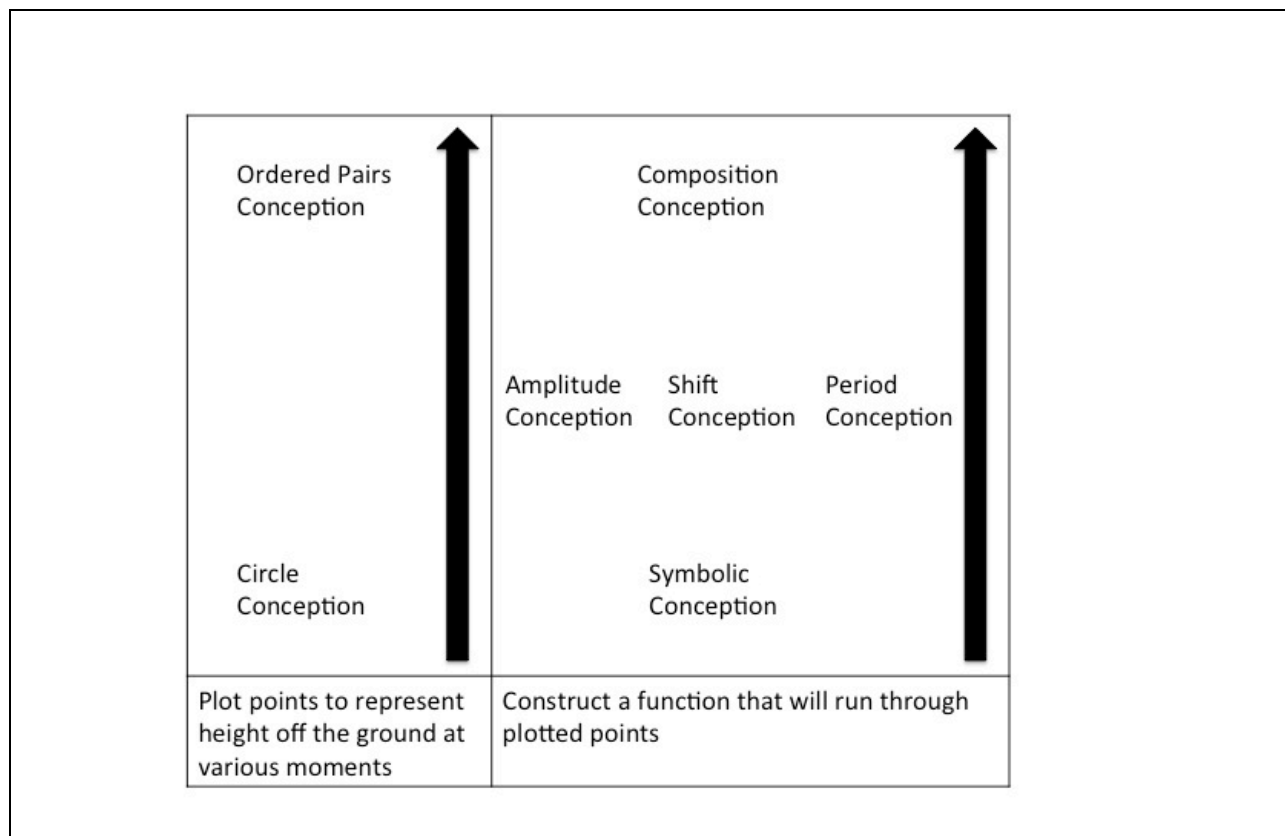


Figure 2.7. The seven conceptions of sine and cosine, organized according to the corresponding problem and the level of viability for solving the problem.

In Figure 2.7 I organized the different conceptions vertically from the least viable conception I identified for solving the given problem, to the most viable conception I identified for solving the given problem. One important note about Figure 2.7 is that I do not make a comparison of conceptions across columns. For example, although the “ordered pairs” conception and the “composition” conception lie on the same horizontal plane, I would not suggest that the two conceptions are equal in terms of their viability. The two conceptions come from two different problems to be solved, and I do not have a precise way to measure across problems. In addition, I would not suggest that the viability of a conception is something that

could be measured according to a precise scale, or that such a scale would be the same across different problems. The organization in Figure 2.7 provides a visual model to distinguish different conceptions. In what follows, I will describe each of the seven conceptions in detail according to the cK ϕ framework, beginning with the two conceptions associated with the first problem.

Circle conception. The first phase of students' work on the problem required them to translate the information provided about their trip on the Ferris wheel to the coordinate plane provided in the Etoys notebook. To do so, students had to plot 4-5 points to represent their height off the ground at various points during the ride on the Ferris wheel. The circle conception of sine and cosine surfaced in students' work while they plotted these points. Students plotted points on the coordinate plane to match the circular motion of the Ferris wheel, rather than using the x -axis to represent time. The circle conception can be described using the cK ϕ model to form the quadruplet $Circ = (P_{Circ}, R_{Circ}, L_{Circ}, \Sigma_{Circ})$ as follows:

Problem (P_{Circ}):

Plot four or five points representing your height off the ground at various moments of riding the Ferris wheel.

Operations (R_{Circ}):

Identify heights at various moments of riding the Ferris wheel (usually minimum height, maximum height, and middle height). Plot the maximum and minimum heights on the y -axis. Plot the middle heights in the first and third quadrant to make a circle.

Representations (L_{Circ}):

Verbal description of trip around Ferris wheel. Virtual representation of Ferris wheel provided in the Etoys notebook. Coordinate plane and discrete points provided in the Etoys notebook.

Control Structure (Σ_{Circ}):

Compare the shape of the plotted points to the shape of the Ferris wheel. If the plotted points make (approximately) a circle, then the points are correct. Check that all of the plotted points are above the x -axis, since height off the ground will always be positive.

Maggie and Cara provided an example of the circle conception upon beginning their work on the problem. In the segment that follows in Table 2.5, Maggie and Cara had just finished reading the problem and were beginning to plot points in the Etoys notebook.

Table 2.5

A Conversation Between Maggie and Cara Invoking a Circle Conception

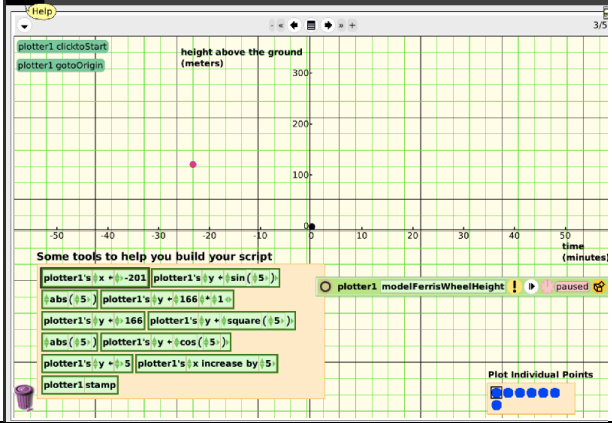
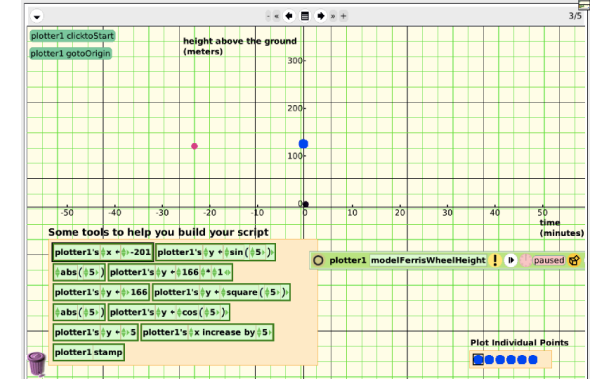
Turn	Speaker	Turn	Screen Work
1	Maggie	All right, so it has a diameter of 130 meters. Okay, so, we should probably know that. I didn't. And the lowest point on the Ferris wheel is 5 meters above. It takes 30 minutes. Okay so, if it takes 30 minutes to go all the way around -	
2	Cara	We have to like, move this.	
3	Maggie	Yeah. That needs to ske-daddle.	
4	Cara	Okay. So our lowest point is 5, right?	
5	Maggie	That's your, 'cuz that's your [inaudible] point. Okay, assuming the London Eye has a diameter of 130 meters. How do we know [pause 2 seconds]? Okay. So go like—We have to, we have to have one at 130. 'Cuz this is the height above the ground in meters. So like, I don't know that looks like about 130 if you ask me.	
6		[Cara plots point at (0, 130).]	

Table 2.5 (cont.)

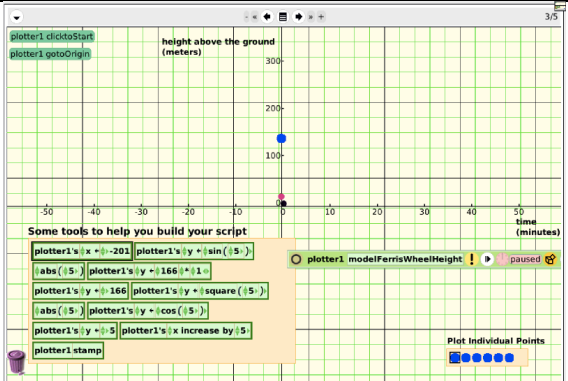
Turn	Speaker	Turn	Screen Work
7	Maggie	And then we have to do one the other way.	
8		[Cara starts dragging points. Doesn't [put anything down.]	
9	Cara	But it said our lowest point was 5. That would be negative [motioning under x -axis].	
10	Maggie	What?	
11	Cara	The lowest point -	
12	Maggie	[Reading] The lowest point on the Ferris wheel is 5 meters above the Th.. river. Oh, so – wait, so we have to start up 5 meters, and then go 130 from there?	
13	Cara	I thought we had to start above, like, even that. Because that's the lowest point, but that doesn't mean... Okay maybe, I don't know. So we have to go here? So where do I go?	
14	Maggie	Go to 5.	
15		[Cara plots a point at (0, 5).]	
16	Maggie	Like right there. Okay, so the lowest point. So it's saying like the lowest point of where the Ferris wheel stops and starts is 5 meters. So now we have to do 130 up from that. Which is where you're at right now. And now we have to do 130 over.	

Table 2.5 (cont.)

Turn	Speaker	Turn	Screen Work
17	Maggie	But we have to figure out how much time it takes , because that's... time's in minutes .	
18		[Pause 3 seconds.]	
19	Cara	30 minutes?	
20	Maggie	Is it 30 minutes divided by...?	
21	Cara	Divided by what? Divided by 3?	
22	Maggie	I was thinking about doing it [pause 2 sec] 30 minutes. So you go, you go, wait. All right so I did 30 minutes times 60 to get – wait, no I don't need to do that. 30 divided by 360. Okay, so wait. So you go 12 minutes each quadrant. Does that work? That sounds right?	
23	Cara	So right here? [Cara puts at point at (12, 0).]	
24	Maggie	Yeah, and then put one...	
25		[Cara puts point at (-12, 0).]	

Table 2.5 (cont.)

Turn	Speaker	Turn	Screen Work
26	Cara	Okay cool. Now what?	
27	Maggie	Now we have to put one... Wait where'd I put the other one?	
28	Cara	It can be a triangle.	
29	Maggie	That's a good – Wait hold on, those have to be level [pointing to points at (12, 0) and (-12, 0)]. Wait, no, those have to be up higher. This is confusing as heck.	
30	Cara	This?	
31	Maggie	Okay go up halfway.	
32		[Cara moves points up to approximately (12, 50) and (12, -50).]	
33	Maggie	There. That looks like a Ferris wheel.	
34	Cara	Why is this not even? It's bothering me.	
35	Maggie	That looks like a Ferris wheel kinda.	

Having just read the description of the Ferris wheel, Maggie and Cara were trying to solve the problem of plotting points to represent their height off the ground at various moments. Maggie and Cara used the operations of plotting the maximum and minimum heights, and then adding points to represent a middle height in order to make the plots into a circle. These operations of plotting the highest and lowest points off the ground were particularly evident in turns 5-6, 9, 12, and 15. Maggie and Cara used the verbal representation of the Ferris wheel scenario to determine where the highest and lowest points would be. They used the discrete points in the Etoys notebook to plot the highest and lowest points on the y -axis. Next, Maggie and Cara did some computations to determine how long they would spend in each quadrant. Maggie's comment that "you go 12 minutes in each quadrant" (turn 22) was evidence that the representation they were constructing was still closely tied to the visualization of the Ferris wheel going around in a circle.

In their next steps (turns 22-25) Maggie and Cara placed two more points, based on their computation, at $(12,0)$ and $(-12,0)$. At this moment (turn 27) Maggie asked where she would put the final point. That question from Maggie indicated that she was using the control structure that all points would need to be above the x -axis. In order to construct a symmetric representation (which looked somewhat circular) Maggie would have needed to place a point below the x -axis. In response to this control structure, Maggie and Cara moved their points from $(12,0)$ and $(-12,0)$ up to $(12,50)$ and $(-12,50)$ (turns 31-32). In turn 31, Cara indicated that putting the points at $y=50$ was not a precise decision, but rather they put the point there because it was approximately halfway between the lowest point and the highest point. In turns 33-35, Maggie and Cara determined that they had satisfied the control structure, "that looks like a Ferris wheel." Although the representation was not a precise circle, it seemed that the plotted points looked

circular enough, and were placed appropriately on the plane, to satisfy Maggie and Cara's conception.

The circle conception was the most common initial conception to surface during students' work on the Ferris wheel problem. Seven of the 11 pairs of students suggested that the plotted points modeling their height off the ground as a function of time should look like a circle. When Ms. Alexander asked Tori and Sean what they expected the graph to look like, Tori immediately responded, "A circle. It has to be a circle, right? 'Cuz it's a Ferris wheel." Most pairs of students moved quickly past the circle conception when Ms. Alexander reminded them that time would need to go on the x -axis of their plot and height on the y -axis. Since the circle conception surfaced early in students' work on the problem, Ms. Alexander was quick to scaffold students past this conception so that they would plot the points correctly. I encouraged Ms. Alexander to allow students to work on the problem as independently as possible, so after this point she gave students participating in the study very few clues about how to solve the problem. Although Ms. Alexander did not give students any more information than what had already been provided in the problem, her comment about the meaning of the two axes seemed to be enough to cue students that the plot of their height off the ground as a function of time would not be represented by a circle.

Calling the circle conception a conception of sine and cosine functions is somewhat counterintuitive, because at the time when students had to plot points to represent their height off the ground, they did not yet necessarily know that the height would be modeled by a sine or cosine function. Still, the circle conception occurred within the sphere of practice of representing real-world phenomena with sine and cosine functions. Translating the circular motion of the Ferris wheel to a periodic function of time is analogous to translating sine and

cosine functions from the unit circle to the Cartesian plane, a translation that has proven to be challenging for students in secondary and post-secondary mathematics classes (Cavanagh, 2008; Weber, 2005; Thompson 2008). Time spent on the Ferris wheel is analogous to degrees around a circle on the unit circle model. Each of those variables acts as an independent variable that must be mapped to the x -axis to represent circular motion as a function. The circle conception that appeared in students' work on the Ferris wheel problem indicated students' struggle to make sense of the independent and dependent variables of the function they would plot. The graphical representation, at least initially, was for students another way to sketch the Ferris wheel. With this conception, the independent variable was represented as degrees around the origin, instead of along the x -axis. An understanding of the relationship between the independent and dependent variables on the Cartesian plane was integral to understand sine and cosine functions.

Ordered pairs conception. The other conception that surfaced during students' work on the first problem was the ordered pairs conception of sine and cosine. With this conception, students used the x -axis of the Cartesian graph to represent time and the y -axis to represent height off the ground. This allowed for students to plot a standard xy -representation of height off the ground as a function of time. The ordered pairs conception can be described in terms of the quadruplet $Ord = (P_{Ord}, R_{Ord}, L_{Ord}, \Sigma_{Ord})$ where:

Problem (P_{Ord}):

Plot four or five points representing your height off the ground at various moments of riding the Ferris wheel.

Operations (R_{Ord}):

Identify the height off the ground at time 0. Identify the highest point off the ground and when it occurs. Divide 30 by 4 to determine the time corresponding to halfway point between the lowest

point and the highest point. Divide 130 by 2 to determine where the halfway point of height occurs. Plot up to five points corresponding to 0 minutes, 7.5 minutes, 15 minutes, 22.5 minutes, and 30 minutes.

Representations (L_{Ord}):

Verbal description of trip around Ferris wheel. Virtual model of Ferris wheel provided in the Etoys notebook. Coordinate plane and discrete provided in the Etoys notebook. The respective meanings of the independent and dependent axes are explicit.

Control Structure (Σ_{Ord}):

The Ferris wheel makes a complete revolution every 30 minutes, so the plot should begin at the lowest point, go up to the highest point, and back down to the lowest point, within the domain from $x=0$ to $x=30$.

The ordered pairs conception of sine and cosine is best illustrated by students' work plotting points in the Etoys notebook. Figure 2.8 illustrates two representations of an ordered pairs conception of sine and cosine from the work of Zach and Jalisa. In the first instance, Zach and Jalisa placed four points at time intervals 0, 10, 20, 30, and 40. The operations Zach and Jalisa used were to place a point at (0,5) to correspond to the lowest point off the ground at time 0. They also placed a point at approximately (40,145) to correspond to the highest point off the ground at time 40. Zach and Jalisa also included points at time 10, 20, and 30 that were approximately evenly spaced between the lowest and highest points off the ground (see the top row of Figure 2.8). Zach and Jalisa used the appropriate maximum and minimum values for their height of the ground, and they established a connection between the time on the Ferris wheel and their height off the ground. At this point, Zach and Jalisa did not make their measure of control explicit, namely that the point corresponding to $x=40$ should be back to the minimum

value of the graph. Zach and Jalisa worked fairly quietly, with Zach controlling most of the work at the computer. Even though they did not make the control structure explicit, Zach and Jalisa had not satisfied the measure of control within this conception, and they continued their work and revised the points.

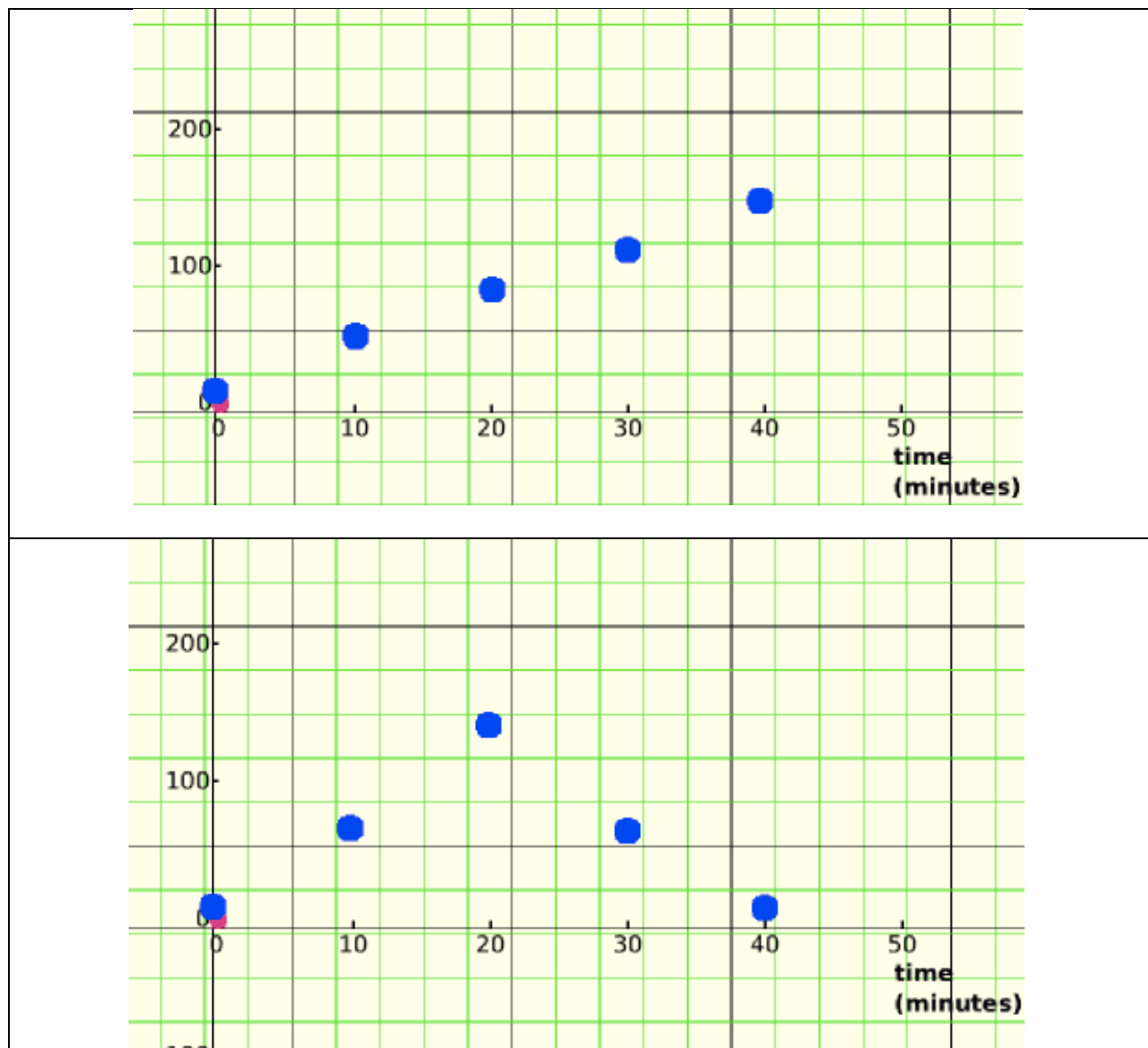


Figure 2.8. Two representations of an ordered pairs conception in the work of Zach and Jalisa.

Zach and Jalisa removed the points they plotted and revised their plot to the representation in the bottom row of Figure 2.8. With the new representation, the points still

indicated an ordered pairs correspondence between time and height off the ground. Zach and Jalisa revised their operations slightly, so that first they plotted points at $x=0$ and $x=40$ indicating the minimum height off the ground. Next, Zach and Jalisa plotted a point at $x=20$ indicating the maximum height off the ground occurring 20 minutes into the ride. Finally, Zach and Jalisa placed points at $x=10$ and $x=30$ that were approximately spaced between the minimum and maximum heights. With these revised operations, Zach and Jalisa were able to satisfy the control structure of the ordered pairs conception, namely that the plot should begin and end at its lowest point. Again, Zach and Jalisa were not especially talkative during their work, and they did not make this measure of control explicit. I inferred, based on the fact that they kept this second set of points and moved on to the next part of the problem, that Zach and Jalisa satisfied the measure of control of the conception.

The problem corresponding to the ordered pairs conception was the same as the problem corresponding to the circle conception of sine and cosine. Also, the representation provided in the set-up of the problem was the same with the two conceptions. However, with the ordered pairs conception, students made the meaning of the x -axis and the y -axis explicit and took those meanings into account to plot the points. Rather than plotting points approximately in the shape of a circle, students first plotted points corresponding to the moment when they would be closest to the ground and the moment when they would be highest in the air (corresponding to 0 minutes and 15 minutes into the ride). Students also did some computations to determine when and where they would be halfway up the ride on the Ferris wheel. The measure of control for the ordered pairs conception was that the Ferris wheel needed to complete one full revolution in 30 minutes (or 40 minutes), so the graph would need to go all the way up and back down between $x=0$ and $x=30$ on the graph.

All of the pairs of students in the study showed evidence of the ordered pairs conception of sine and cosine during their work on the Ferris wheel problem. The ordered pairs conception allowed students to identify the shape of the graph as a sinusoidal curve, which made it possible for students to begin constructing a function that would run through the points. For the seven groups who began with a circle conception of sine and cosine, the ordered pairs conception followed when students realized that the circle conception was not viable for constructing a function to solve the problem. The remaining pairs of students began their work already with a ordered pairs conception of sine and cosine, and those groups never displayed any evidence of a circle conception.

Symbolic conception. The symbolic conception, and all following conceptions, surfaced during students' work on the problem to construct a function that would overlap the points that they had plotted. I refer to this first conception as a symbolic conception, because students drew on their prior knowledge of cosine functions expressed symbolically as $f(x) = a\cos(bx) + c$ in standard algebraic notation⁹. The symbolic conception of sine and cosine function,

$Sym = (P_{Sym}, R_{Sym}, L_{Sym}, \Sigma_{Sym})$, can be expressed with the following:

Problem (P_{Sym}):

Write a function that looks like $f(x) = a\cos(bx) + c$.

Operations (R_{Sym}):

Put a number in front of the cosine function, inside of the cosine function in front of the independent variable, and at the end of the expression. The numbers should be related to the

⁹ Ten of the eleven pairs of students used a cosine function to model the points. One pair used a sine function with a horizontal shift. I use cosine functions throughout, but a sine function could be used interchangeably.

numbers given in the set-up of the problem. The operations for determining the values of a , b , and c , may vary.

Representations (L_{Sym}):

Example of Etoys script (which included an example of how to put together a quadratic function). Drag-and-drop tiles to select functions, operators, and values to use to compose a function. Written symbolic representation of sine or cosine function.

Control Structure (Σ_{Sym}):

Function should look like $f(x)=a\cos(bx)+c$. If the components of the function are not in that order, the function is not correct. The resulting function should plot a sinusoidal curve.

The example of Tori and Sean from Tables 2.3 and 2.4 illustrates the symbolic conception that surfaced in students' work. In that example, Sean was having difficulty putting a number in front of his cosine function. Sean attempted to use an operation of putting a number in front of the cosine function to solve the problem of writing a cosine function to look a certain way. The symbolic conception of sine and cosine functions emerged during students' work on the second problem, meaning that they had already plotted a collection of discrete points to represent their height off the ground at various moments. However, students did not use the representation provided by the plotted points with this conception. Instead, they relied primarily on their prior knowledge of what a symbolic representation of a sine or cosine function should look like. Once students recognized that they would need to use a sinusoidal function, students' perception of the problem was to write a function that looked like the previous sine and cosine functions they had written. The operations to solve this problem were to place a number in front of the cosine function, inside the cosine function, and at the end of the function. Since students were working with Etoys, the operations included dragging and dropping the tiles in the

notebook. The representations for the symbolic conception were the drag and drop tiles provided in Etoys. In addition, students relied on their prior knowledge of the standard symbolic representation of a cosine function as $f(x)=a\cos(bx)+c$. Students attempted to use the drag-and-drop tiles provided by Etoys to construct a representation consistent with standard symbolic representations.

With the symbolic conception, students' use of Etoys became very prominent, primarily because the syntax requirements of Etoys served to work *against* students' symbolic conceptions. The syntax requirements of Etoys made it impossible to write a function such as $f(x)=a\cos(bx)+c$ without indicating any mathematical operations (e.g., the implied multiplication between a and $\cos(bx)$) or providing specific values. In other words, using the drag-and-drop tool in Etoys prevented students from achieving a control structure for the symbolic conception. Using the drag-and-drop tool in Etoys, students were unable to construct a symbolic representation consistent with the previous symbolic representations they had used. In this way, students' conceptions based on their prior knowledge of sine and cosine functions were not viable for their work on the Ferris wheel problem. This condition provoked students to think about sine and cosine functions in new ways, giving more attention to the relationship between the symbolic representation of the function and the shape of the graph that it would plot.

Amplitude conception. A more viable conception than the symbolic conception for working on the problem of constructing a function that would run through points representing one's height off the ground was the amplitude conception. The amplitude conception emerged after students had come up with an initial plot of a sine or cosine function. The amplitude conception of sine and cosine functions, $Amp = (P_{Amp}, R_{Amp}, L_{Amp}, \Sigma_{Amp})$, can be expressed with the following:

Problem (P_{Amp}):

Given the graph of a sinusoidal function and a set of discrete points, stretch the graph vertically so that the graph reaches the maximum and minimum points.

Operations (R_{Amp}):

Experiment by changing the numbers in the previously constructed cosine function. Pay attention to how the changes affect the vertical stretch of the graph. Change the coefficient of the cosine function from a positive to a negative number to examine what makes the graph reach higher and lower. Examine changes as the coefficient of cosine gets larger or smaller. Add a larger or smaller value at the end of the cosine expression to make the graph reach higher or lower. Change the value of the “x increase by” tile to make the graph coincide with the discrete points previously plotted.

Representations (L_{Amp}):

Previously plotted points representing height off the ground at various times. Drag-and-drop tiles to select functions, operators, and values to use to compose a function. Plotted graphs.

Control Structure (Σ_{Amp}):

The obtained graph should stretch as high as the highest plotted point and as low as the lowest plotted point. The graph should go through at least some of the plotted points, though not necessarily all.

Once students had constructed an initial cosine plot, the problem was to stretch the plot vertically so that it would stretch as high and as low as the previously plotted discrete points. Using the drag-and-drop tiles provided by the Etoys notebook, students experimented with the different values composing the sine or cosine function to examine how they could stretch the graph vertically. The measure of control for the conception was that the graph should reach the

highest and lowest plotted points. In addition, students used as a measure of control that the graph should run through at least some of the plotted points, though not necessarily all.

On the second day of the lesson, Sean, Tori, and Aubrey illustrated an example of the amplitude conception as they tried to adjust the coefficient of their cosine function to stretch vertically to the appropriate maximum and minimum values. The students debated what should be the maximum value of the graph, and they indicated that they had established a connection between the coefficient of the cosine function and the vertical stretch of the graph.

Table 2.6

A Conversation Between Tori, Sean, and Aubrey Invoking an Amplitude Conception

Turn	Speaker	Turn	Screen Work
83	Tori	The maximum is seven – The maximum is 140, right?	
84	Aubrey	Mm hmm.	
85	Sean	The maximum is 140, but that's wrong.	
86	Tori	No.	
87	Sean	That'd be a maximum of 70.	
88	Aubrey	True.	
89	Sean	So if half the max, half of seventy is thirty – So we have 35. Try it.	
90		[Tori changes the coefficient of the cosine function to -35 and runs the script.]	
91	Tori	[Sigh.] Wait, so we did this wrong then.	
92	Sean	Did what wrong?	
93	Tori	Where's the paper from yesterday? See? Cuz we said it was 70 here.	
94	Sean	We do have it right. You're right.	
95	Tori	Like we said that was 70.	
96	Sean	Yeah, you're fine.	
97	Tori	Wouldn't the highest point be 145 , though? If the Eye has a diameter of 145 meters? If you're at the top, you're at the diameter.	
98	Sean	Well, we're just drawing our graph not to scale.	

In turn 83, Tori noted that the maximum value of their graph should be at 140 (actually it should have been at 145). Sean argued that actually the maximum value of the graph should be at 70 (turns 85-87). It is not clear why Sean decided that their graph should reach a maximum height of 70, instead of the maximum height of 145 that they had previously plotted with the blue points. Regardless, the conversation and actions that followed indicated that Tori, Sean, and Aubrey established connections between the coefficient of the cosine function and the vertical stretch of the graph. In turn 89, Sean specifically noted that they should change the coefficient of the function to -35 in order to make the graph reach a maximum of 70. In the following turn, he made this adjustment, and the members of the group could observe that the graph of the new function reach a maximum value of 70. After this, Tori and Sean had a discussion about whether the coefficient they had previously determined had been incorrect (turns 91-96).

Although the conception in Table 2.6 had not yet elicited a correct solution to the problem, Tori and Sean again indicated in turns 97 and 98 that they were working with an amplitude conception of sine and cosine functions. Specifically, Tori made another comment in turn 97 that the graph should stretch up to a maximum of 145. This comment suggests that Tori was paying attention to the change in the graph as a result of their change to the syntax. She noted that the graph was not reaching a high enough point. Moreover, Tori seemed to want to adjust the syntax to make the graph reach the appropriate maximum height. Sean, even though his idea for a coefficient of -35 was incorrect, still indicated that he was working with an amplitude conception. Sean commented, “we’re just not drawing our graph to scale” in turn 98. With that comment, Sean indicated that he recognized that the vertical stretch of the graph was connected with the coefficient of the function. He had made a choice to use a coefficient that would draw the graph “not to scale.” The students seemed satisfied with their solution,

potentially because they had satisfied the control structure that the graph would overlap at least some of the blue points they had previously plotted.

It is important to note that the amplitude conception did not necessarily result in students finding the correct coefficient of the cosine function to model the given context. The main reason for this was that, with the amplitude conception, students paid attention to the amplitude of the function but not the vertical shift or period. In the example above, Tori, Sean, and Aubrey may have been able to better resolve their disagreement had they given consideration to the vertical shift of the graph. The amplitude conception was tantamount to considering cosine as a function whose only parameter was amplitude. This most often led to an incorrect coefficient for the cosine function, particularly if students had not considered the vertical shift of the function prior to considering the amplitude. Still, the amplitude conception indicated a move away from a symbolic conception towards making a connection between the symbolic and graphical representations of the cosine function.

Shift conception. The shift conception of sine and cosine functions is analogous to the amplitude conception, except that with the shift conception students considered the vertical shift in isolation of the other parameters affecting the shape of the graph of a cosine function. The shift conception of sine and cosine function, $Shift = (P_{Shift}, R_{Shift}, L_{Shift}, \Sigma_{Shift})$, can be expressed with the following:

Problem (P_{Shift}):

Given the graph of a sinusoidal function and a set of discrete points, shift the graph up or down so that it goes through the maximum and minimum points.

Operations (R_{Shift}):

Change numbers in the previously constructed cosine function. Test whether changing the coefficient of the cosine function will move the graph up or down. Add a larger or smaller value at the end of the cosine expression to make the graph reach higher or lower. Change the value of the “x-increase-by” tile to make the graph coincide with the discrete points previously plotted.

Representations (L_{Shift}):

Previously plotted points representing height off the ground at various times. Drag-and-drop tiles to select functions, operators, and values to use to compose a function. Plotted graphs.

Control Structure (Σ_{Shift}):

The obtained graph should go through the maximum and minimum plotted point. The graph should go through at least some of the plotted points, though not necessarily all.

The operations and representations of the shift conceptions were similar to the amplitude conception, although the problem that students worked to solve was different. Whereas with the amplitude conception students tried to solve the problem of stretching the graph out, with the shift conception students had to solve the problem of moving the graph up or down. Still, the operations they used to do this were to experiment with the different parameters of the cosine function. Similarly to the amplitude conception, the control structure of the shift conception was that the graph should go through the maximum and minimum points that had been previously plotted. This control structure for both the amplitude and shift conceptions are very similar due to the fact that both the amplitude and the vertical shift of a sine or cosine function will impact the maximum and minimum values of the function.

Period conception. There was a third conception I identified in students’ work that was analogous, in terms of viability for solving the problem, to the amplitude and shift conceptions.

This was the period conception. The period conception of sine and cosine function,

$Per = (P_{Per}, R_{Per}, L_{Per}, \Sigma_{Per})$, can be expressed with the following:

Problem (P_{Per}):

Given the graph of a sinusoidal function and a set of discrete points, stretch or shrink the graph horizontally so that it goes through one period after 30 minutes on the x -axis.

Operations (R_{Per}):

Experiment by changing the numbers in the previously constructed cosine function. Test bigger or smaller numbers inside the cosine function. Since the period of the graph should be 30, try making the coefficient of x equal to 30 inside the cosine function. Check computations from a previously known formula, $Period = \frac{2\pi}{b}$, to determine the coefficient b inside the cosine function. Change the value of the “ x -increase-by” tile to make the graph coincide with the discrete points previously plotted.

Representations (L_{Per}):

Previously plotted points representing height off the ground at various times. Drag-and-drop tiles to select functions, operators, and values to use to compose a function. Plotted graphs.

Control Structure (Σ_{Per}):

The resulting graph should go through one complete period from $x=0$ to $x=30$. The graph should go through at least some of the plotted points, though not necessarily all.

An example from the work of Mitchell and Reese illustrates the period conception of sine and cosine functions. In Table 2.7, Mitchell and Reese examined how to construct a coefficient of the independent variable inside the sine function. Mitchell and Reese were working on the version of the Ferris wheel problem in which the diameter of the Ferris wheel was 130 meters, and it took 30 minutes to complete one revolution. Mitchell and Reese were using a sine

function to represent their height off the ground as they were riding the Ferris wheel. At the start of the transcript in Table 2.7, Mitchell and Reese were using the function $65\sin(30x) + 65$.

Table 2.7

An Example of a Period Conception From the Work of Mitchell and Reese

Line	Speaker	Turn	Screen Work and Written Work
1	Mitchell	Let's see this. So x , we need to make the period bigger . So it's gotta be more.	
2	Reese	Yeah and you have to – The numbers have to be farther apart.	
3	Mitchell	What do we need the period to be? Thirty , so it needs to be -	
4	Reese	Two pi over 30. Pi over 15.	
5	Mitchell	Two pi over b equals 30.	$\frac{2\pi}{b} = 30$
6	Reese	Oh.	
7	Mitchell	Thirty b equals two pi. Two pi over thirty. So the period has to be 2 pi over 30. And how are we supposed to put that in this?	$2\pi = 30b$ $\frac{2\pi}{30} = b$
8		[Students enter the coefficient of the x inside the sine function and run the code.]	

Mitchell and Reese's work in turn 1 of the transcript indicates the initial operation that Mitchell and Reese used to stretch their graph horizontally. Since the period of the sine graph needed to be 30, Mitchell and Reese used the number 30 as the coefficient of the independent variable inside the sine function. With his comment in turn 1, Mitchell revealed the problem he was trying to solve, to "make the period bigger." In turn 2 Reese suggested they would need to move the numbers farther apart, indicating an attempt to perform the operation of changing the "x-increase-by" value to spread the stamped points farther apart horizontally. However, instead of performing that operation, in turns 3-7 Mitchell and Reese used the formula they had previously learned for computing the coefficient of the independent variable to account properly for the period. The result, indicated by the screen work in turn 8, was a graph that was stretched horizontally to have a period of 30 minutes.

The period conception of sine and cosine allowed students to focus on how stretched or compressed their graphs would be horizontally. Similar to the amplitude and shift conceptions, students experimented with how changing the values within the cosine function with drag-and-drop tiles would change the period of the function. Although students knew the period of the function, and they had previously used the formula $b = \frac{2\pi}{P}$, where P represents the period, they generally did not use this method for computing the coefficient of x inside the cosine function. Students often began by setting the coefficient of x equal to the period of the function, and they experimented with increasing or decreasing the coefficient from there to make the graph stretch appropriately.

Composition conception. The composition conception of sine and cosine was the most viable conception for students to solve the Ferris wheel problem. With the previous three conceptions students considered amplitude, shift, and period in isolation from one another, and

they focused on different parameters of the function in isolation. With the composition conception, students' conceptions of sine and cosine functions reflected an understanding of sinusoidal functions as a composition of functions satisfying an interrelated set of parameters.

The composition conception of sine and cosine functions, $Comp = (P_{Comp}, R_{Comp}, L_{Comp}, \Sigma_{Comp})$, can be expressed with the following:

Problem (P_{Comp}):

Given a set of parameters, and points representing the shape of a graph, compose a sinusoidal function so that the amplitude, shift, and periodicity of the function satisfy the parameters.

Operations (R_{Comp}):

Determine the highest and lowest points of the graph. Divide the difference by two to determine the coefficient of the cosine function. Since the plotted points begin at their minimum point, make the coefficient of the cosine function negative. Apply the formula $b = \frac{2\pi}{P}$, where P corresponds to the revolution time of the Ferris wheel, to determine the coefficient of x inside the cosine function. Once the amplitude is determined, add a value at the end to shift the graph up so that the minimum value of the function is 5.

Representations (L_{Comp}):

Previously plotted points representing height off the ground at various times. Drag-and-drop tiles to select functions, operators, and values to use to compose a function. Written computations for values of parameters. Plotted graphs.

Control Structure (Σ_{Comp}):

The resulting graph of the function should complete one period between $x=0$ and $x=30$. The graph of the function should (approximately) run through the previously plotted discrete points.

Verify with partner or previous notes whether procedures for finding values in functions are consistent with prior work.

The composition conception of sine and cosine functions was the most sophisticated conception that emerged in students' work. It was the conception that was aligned with Ms. Alexander's goals for students to learn about sine and cosine in the context of the problem. The composition conception of sine and cosine functions was the only conception in which students used systematic procedures, aligned with what they had previously studied about sine and cosine functions, to determine the values of the various quantities composing the cosine function. In addition, this conception allowed students to make connections between the Ferris wheel context of the problem, the symbolic representations they constructed with the Etoys syntax, and the graphical representation their syntax produced.

Students' control structures within the composition conception of sine and cosine functions indicated an important change from the previous conceptions. With the previous three conceptions of sine and cosine functions, students primary measure of control was to check whether the graph they plotted ran through the collection of discrete points that they had previously plotted, or whether it went through a subset of those points. Within the composition conception, students checked that their plotted functions ran approximately through the points that they had previously plotted. Using systematic methods for computing the values composing the function gave students a more secure measure of control than checking whether the plotted graph ran exactly through the previously plotted points, which themselves had been approximately placed. Instead, students checked that their methods for computing the values in the function were consistent with the information provided in the problem and the methods they had previously established for constructing sine and cosine functions.

The Evolution of Students' Conceptions of Sine and Cosine Functions

After identifying what conceptions of sine and cosine functions surfaced during students' work on the Ferris wheel problem, I sought to answer my second research question: *How did students' conceptions of sine and cosine evolve through their work with their partners?* Table 2.8 summarizes the movements among conceptions of each pair of students. The arrows in Table 2.8 indicate a movement from one conception to another. The table reflects that eight of the groups in the study began with a circle conception and then shifted to an ordered pairs conception. Three of the groups began their work with an ordered pairs conception. After the ordered pairs conception, students varied in their movements among conceptions. Although Ms. Alexander gave students the solution to the Ferris wheel problem during her wrap-up of the lesson on Day 2, not all students achieved a composition conception of sine and cosine functions during the time spent working with their partner. There were four pairs that did not invoke a composition conception, and there were seven pairs that did invoke a composition conception.

Table 2.8

Movements Among Conceptions of Each Pair of Students in the Etoys Study

Pair	Movements Among Conceptions
Carson Abbey	Circle → Ordered Pairs → Symbolic → Amplitude → Symbolic → Amplitude → Shift → Composition
Bailey Aubrey	Circle → Ordered Pairs → Amplitude → Symbolic → Shift → Period → Composition
Jalisa Zach	Ordered Pairs → Composition
Cara Maggie	Circle → Ordered Pairs → Amplitude → Shift → Period → Amplitude → Shift → Period
Gia Courtney	Circle → Ordered Pairs → Symbolic → Amplitude → Symbolic → Shift → Period
Mike Jessa	Ordered Pairs → Symbolic → Shift → Amplitude → Shift → Period → Composition
Hannah Dayana	Circle → Ordered Pairs → Amplitude → Period → Shift → Period → Amplitude
Mitchell Reese	Ordered Pairs → Composition → Period → Composition
Shane Maya	Circle → Ordered Pairs → Symbolic → Amplitude → Symbolic → Shift
Tori Sean	Circle → Ordered Pairs → Symbolic → Amplitude → Shift → Amplitude → Shift → Period → Composition
Lucas Elizabeth Andy	Circle → Ordered Pairs → Symbolic → Amplitude → Period → Composition

During their work on the problem, students made what I call forward movements and lateral movements between conceptions of sine and cosine functions. Forward movements were movements from conceptions that were less viable towards conceptions that were more viable for solving the Ferris wheel problem. Lateral movements were movements between different—

but equally viable—conceptions. To make sense of students’ forward and lateral movements between conceptions, I used the hierarchy of conceptions that I presented in Figure 2.7. During students’ work on the first problem, plotting points to represent their height off the ground at a given time, students either began with a circle conception or with an ordered pairs conception. All of the pairs who began with a circle conception shifted to an ordered pairs conception of sine and cosine.

While students worked on the second problem, the symbolic conception of sine and cosine functions was the least viable for solving the problem. I identified the symbolic conception as the least viable because, with this conception, students did not make connections between the syntax representation and the graphical representation created by the output of their syntax in Etoys. Students relied only on their prior knowledge of how a symbolic representation of sine or cosine should look. Since a solution to the Ferris wheel problem required some connection between the different representations, the symbolic conception was not viable for solving the problem. Six of the 11 pairs of students invoked a symbolic conception of sine and cosine as the first conception for working on the problem of constructing a function. Seven pairs of students showed evidence of a symbolic conception at some point during their work.

More viable than the symbolic conception, the amplitude, shift, and period conceptions occupied the same status in the hierarchy of conceptions that I identified. I put those three conceptions on the same horizontal plane because the amplitude, shift, and period conceptions were very similar in terms of their respective operations, representations, and control structures. The primary difference between the three conceptions was the nature of the problem, and which parameter of sine or cosine was allowed to vary while all other remained constant. Each of the amplitude, shift, and period conceptions was limited in that, with each of these conceptions,

students paid attention to only one aspect of the cosine functions. To have a more sophisticated conception of sine and cosine, students would need to pay attention to all of the interrelated parameters that work together to compose a sine or cosine function. Given the shortcomings of the amplitude, shift, and period conceptions, these conceptions had one major advantage over the symbolic conception that made them more viable for solving the Ferris wheel problem. Specifically, with the amplitude, shift, and period conceptions, students attended to the connections between the syntax representation and the graphical representation in Etoys. Making this connection was critical for solving the Ferris wheel problem, so these conceptions indicated a movement forward in students' understanding of sine and cosine functions.

The most viable conception of sine and cosine functions for solving the Ferris wheel problem was the composition conception. The composition conception provided the most holistic understanding of sine and cosine functions, as functions with multiple parameters all acting together to create graphs with various properties. Similarly to the previous three conceptions, students with a composition conception attended to the connections between syntax and graphical representations in Etoys. With the composition conception, students also attended to the relationships between the parameters *within* the Etoys syntax, specifically the parameters accounting for the amplitude and vertical shift of the cosine function. In addition, students made connections between the real world context, the Etoys syntax, and the graphical output of the syntax. Accounting for these three different representations of the problem allowed students to have a systematic way to solve the problem.

Figure 2.8 provides a prototype of the most common path through which students moved through the conceptions of sine and cosine. Figure 2.8 starts with a circle conception, because most students began their work on the Ferris wheel problem with a circle conception of sine and

cosine. After the circle conception, students moved to an ordered pairs conception of sine and cosine, which was the most viable conception for plotting points to represent one's height off the ground.

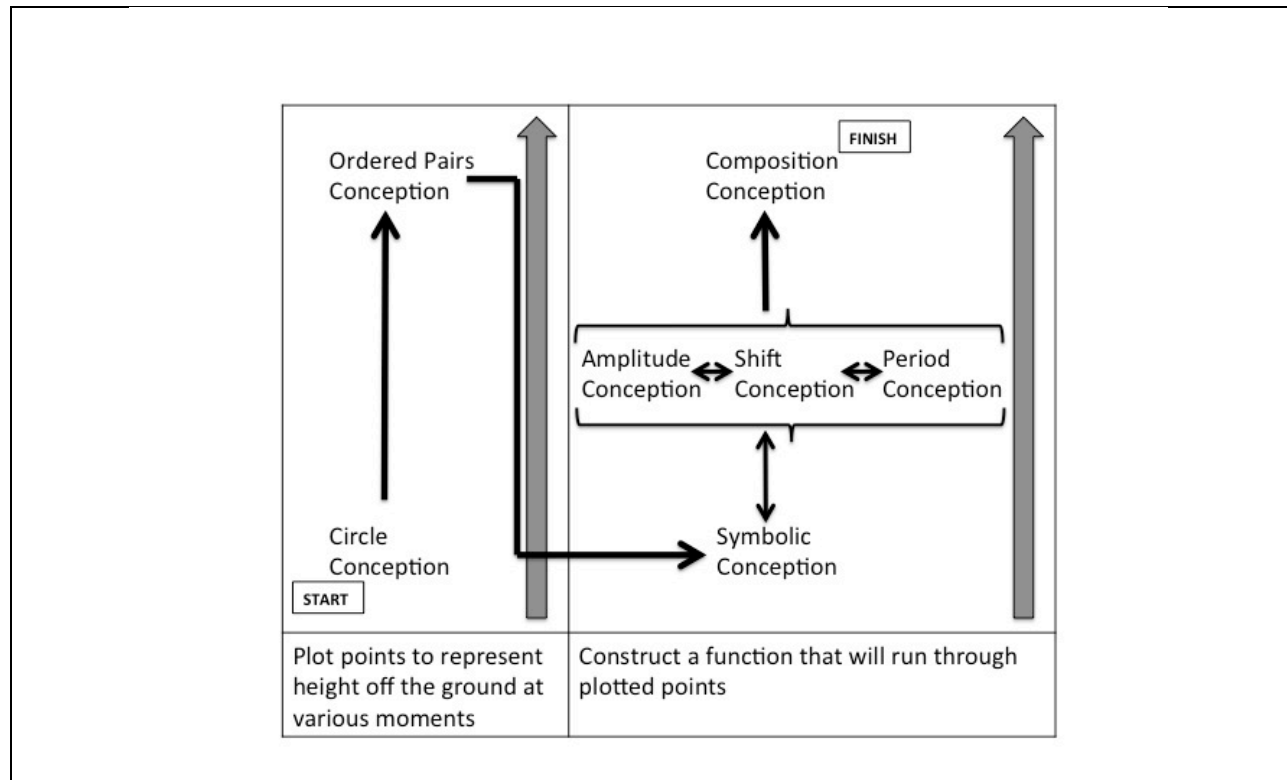


Figure 2.9. A prototype of students' movements through the seven different conceptions of sine and cosine.

As students worked on the second sub-problem of the Ferris wheel problem, I identified more variation in students' conceptions. The prototype in Figure 2.8 indicates a movement from an ordered pairs conception to a symbolic conception, because most groups began their work on the second problem with a symbolic conception of sine and cosine. After the symbolic conception, students shifted back and forth between the amplitude, shift, and period conceptions of sine and cosine functions. In addition, four pairs of students (Carson and Abbey, Bailey and Aubrey, Gia and Courtney, and Shane and Maya) reverted back to a symbolic conception after moving on to an amplitude, shift, or period conception of sine and cosine. The two-way, vertical

arrow between the symbolic conception and the three mid-level conceptions indicates that, in a prototypical case, a pair of students may likely move forward and backward between those conceptions even as they move laterally among the three conceptions.

The final conception that students displayed was the composition conception. The composition conception allowed students to solve the Ferris wheel problem. Only one pair of students—Mitchell and Reese—moved backwards from a composition to a period conception. The backwards move seemed to be provoked when Mitchell's strategy for computing the coefficient of the independent variable failed, so Mitchell and Reese began to examine specifically the period of the function. The average number of conceptions students displayed after the order pairs conception but before the final composition conception was 4. This indicates that students invoked around 4 different conceptions after the work of plotting their points, but before achieving a composition conception. These included the lateral movements between amplitude, period, and shift conceptions, as well as occasional movements backwards to symbolic conceptions. After making lateral shifts between the three conceptions, students moved forward to a composition conception of sine and cosine functions.

Tori and Sean gave an example of movement through all of the different conceptions of sine and cosine. Tori and Sean began on day 1 of the lesson with a circle conception and then moved forward to an ordered pairs conception of sine and cosine. Once they had plotted their points, Tori and Sean initially invoked a symbolic conception. After that, Tori and Sean moved forward to an amplitude conception of sine and cosine functions, which is where they finished the first day of work on the problem. On the second day, Tori and Sean picked back up with an amplitude conception of sine and cosine. They moved laterally to a shift conception, then back to amplitude, back to shift, and over to a period conception over the course of their work on day

2. Near the end of day 2, Tori and Sean invoked a composition conception of sine and cosine functions, which enabled them to solve the Ferris wheel problem.

One other path through the conceptions of sine and cosine emerged in the work of two pairs of students during the Ferris wheel lesson. Two pairs of students—Mitchell and Reese, and Zach and Jalisa—moved through the more direct path outlined in Figure 2.9. The two pairs of students began with an ordered pairs conception during their work on the first problem, and they moved directly to a composition conception during their work on the second problem. Mitchell and Reese were the only pair to use a sine function to represent the Ferris wheel situation. They used the Etoys notebook to construct and graph their function, but they did not move between the less viable conceptions before moving to the composition conception. Although Mitchell and Reese shifted to a period conception briefly towards the end of their work, the majority of Mitchell and Reese's work was situated within a composition conception. Zach and Jalisa immediately showed evidence of a composition conception of sine and cosine functions, but after plotting their points they constructed the appropriate function with paper and pencil, not using Etoys. Once they had already determined the appropriate function, Zach and Jalisa experimented with Etoys to input their cosine function into their script. Although they were still attempting to learn to use Etoys, Zach and Jalisa's conception of sine and cosine was not affected by this experimenting.

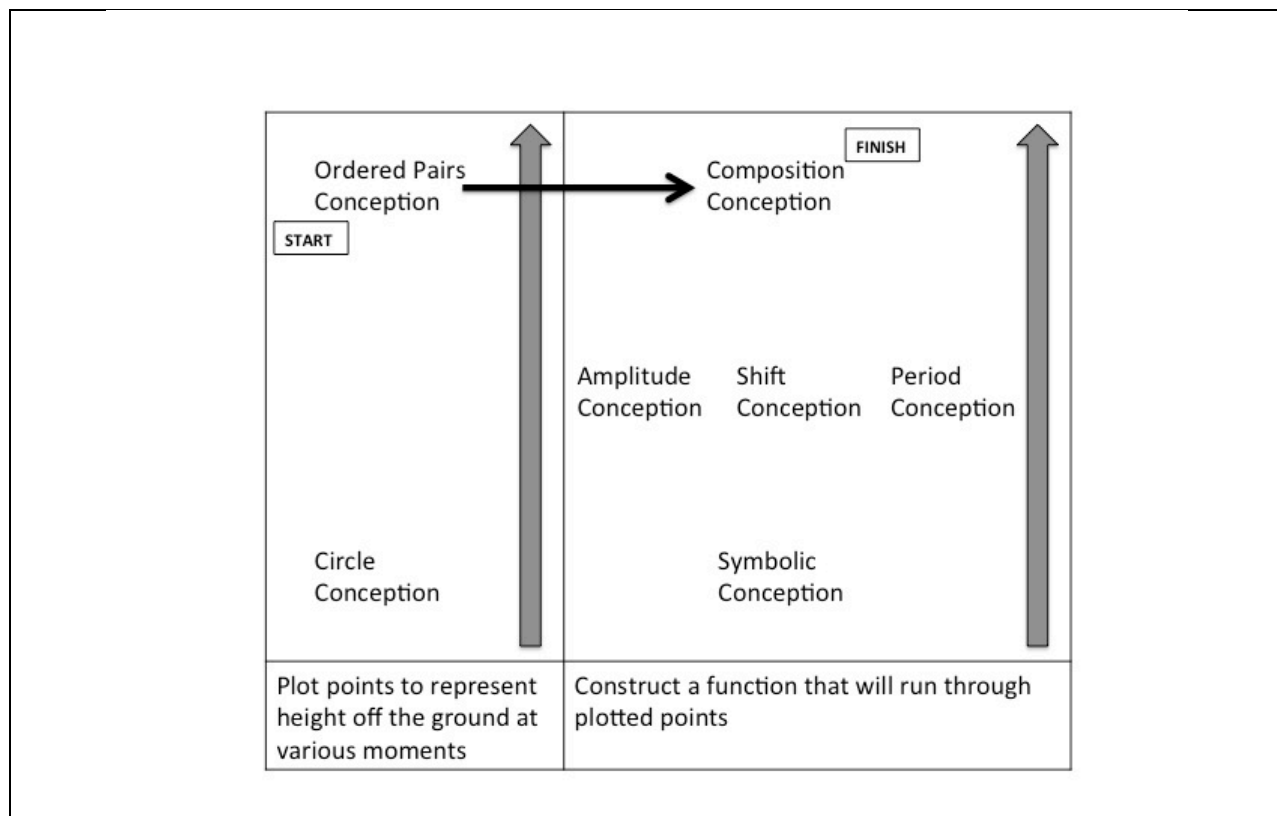


Figure 2.10. A prototype of students' movements through two different conceptions of sine and cosine.

Students' Use of Etoys in Their Conceptions of Sine and Cosine

The final step of this study was to answer my third research question: *How did students use the tools of Etoys in their conceptions of sine and cosine?* There were three main tools in Etoys that students appropriated into their work on the Ferris wheel problem. Those tools, which are summarized in Table 2.9, include the *drag-and-drop* tool, the *scanning* tool, and the *x-increase-by* tool. The drag-and-drop tool was the tool in Etoys that allowed students to drag tiles to construct Etoys scripts. Rather than inputting commands through the keyboard to write a script, students could piece together a script by dragging tiles with the mouse and placing them together.

Table 2.9
Summary of Students' Use of Tools in Etoys

Tool	Description of Students' Use
Drag-and-drop tool	Students dragged tiles in Etoys to construct scripts. Rather than inputting commands from scratch, students could piece together commands from tiles that were available.
Scanning tool	Students used the increase and decrease buttons in the Etoys syntax to change numbers incrementally. Scanning could be <i>deliberate</i> , where students had some idea of which direction or how much to scan. Or scanning could be <i>spontaneous</i> , with no clear idea of how much or which direction to scan.
x -increase-by tool	Students could change the value that the x -coordinate of the plotter would increase by between iterations of the script. When the value of “ x increase by” was very large, plots would be spaced far apart. When the value of “ x increase by” was relatively small, plots would be spaced close together.

Drag-and-drop tool. The first especially significant aspect of students' use of the drag-and-drop tool was that it provoked students to move beyond a symbolic conception of sine and cosine functions. The reason for this is that the syntax requirements of the Etoys notebook would not allow students to put together a function in the way that they wanted with a symbolic conception. To construct their Etoys scripts, students first selected the trigonometric function they wanted to use, and then filled in the other components of the function piece by piece. An expression like $a \cos(x)$ would most easily be entered into Etoys like $\cos(x) \times a$. The difference between having an a in front of the function or at the end of the function may seem to an observer like a minor feature of style. But for students, being provoked to make explicit the multiplicative relationship between the parameter a and the function $\cos(x)$ made it not viable to use a symbolic conception of sine and cosine functions.

In addition to pushing students to make mathematical operations explicit, the drag-and-drop tool provoked students to consider the meaning of each component of a sine or cosine

function. This was an issue specifically in determining what would go inside a sine or cosine function. When selecting a cosine function in Etoys, the drag-and drop tiles by default give the cosine of 5, so the first tile that students selected would be $\cos(5)$ (see Figure 2.10). With a symbolic conception of sine and cosine functions, this was not immediately problematic to students. Having something inside the cosine function, even if it was not a variable, seemed to be enough to make the function look like a cosine function. It only became problematic when students attempted to plot their function, and they found that they were plotting a constant function. Using the drag-and-drop tool to build their functions made the symbolic conception not a viable way to solve the problem, because they could not satisfy the measure of control that a sine or cosine function should make a sinusoidal curve. In this way, working within the syntax requirements of the Etoys notebook provoked students to move beyond a strictly symbolic conception of sine and cosine. They had to pay more attention to the meanings of the components of the functions and the relationships among those components.

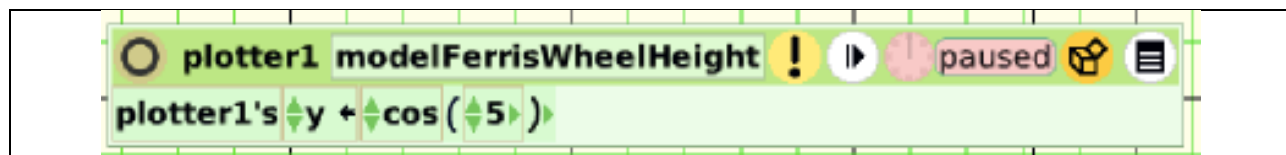


Figure 2.11. With the drag-and-drop tool, the cosine function by default took an input of 5. Students needed to drag a tile with an independent variable into the cosine function.

Scanning tool. The scanning tool in Etoys allowed students to change the values of numbers, or change functions, once they had placed a tile in a script (Figure 2.11). The scanning tool was an alternative to directly entering a value or a function by typing it into the script. By scanning, students could experiment with a range of numbers to examine the outputs of the script. Students' use of the scanning tool could either be *deliberate* scanning or *spontaneous* scanning. The different ways that students used the scanning tool in Etoys is analogous to the

ways that researchers have identified students' use of the dragging tool in dynamic geometry environments (Arzarello et al., 2002; Hollebrands, 2007). With deliberate scanning, students indicated some strategy for which direction or how much they would scan. For example, deliberate scanning could occur if a student recognized that the period of a given graph was too large, so the coefficient of the x -variable inside the function should be increased. With spontaneous scanning, students changed the values of the numbers in their scripts with no indication about why or how much they should change. It is possible that, even in cases that appeared to be spontaneous scanning, students may have made some deliberate choice that they did not make explicit. The conversations among partners during pair work helped provoke students to explain whether or not they were using the scanning tool in a deliberate way.

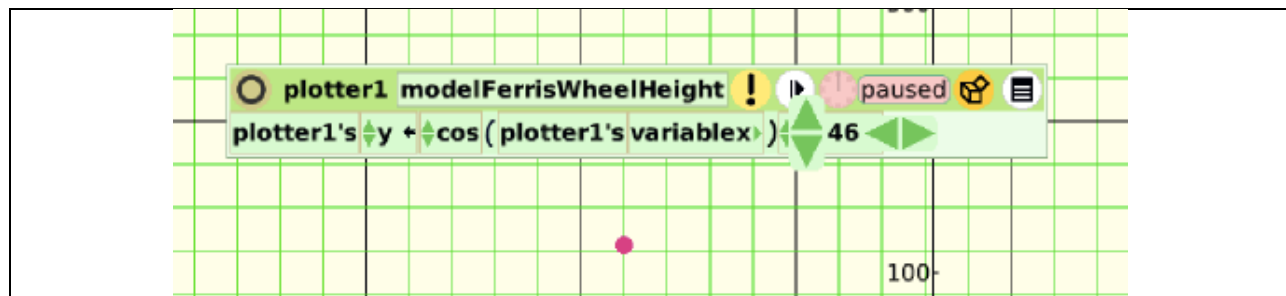


Figure 2.12. Students used the scanning tool, by clicking on the arrows in the script, to change the values in their functions incrementally.

As students moved between the amplitude, shift, and period conceptions of sine and cosine, they used the scanning tool to experiment with the different parameters of the cosine function (see Figure 2.11). Each of those three conceptions began with an already constructed cosine function as the basis for students' work. From that graph, students used the scanning tool to change the numbers and operations within their functions in order to stretch, compress, or shift their graphs in a way that would make them go through the discrete points that they had previously plotted. Using the scanning tool to experiment with the values of the cosine function did not always provide students a systematic, or precise, way to determine the necessary values

of their functions. Especially when students scanned spontaneously, they did not always attend to the connections between the inputs and the outputs of their script to get precise values in their syntax. However, when students scanned deliberately, this process allowed students to examine the relationship between the symbolic representation of the function and the shape of the graph that it made.

When students used the scanning tool, the changes in the outputs of their scripts promoted lateral movements between amplitude and shift conception of sine and cosine functions. With the amplitude conception, students held all parameters of a cosine function fixed except for the vertical stretch of the graph. With the shift conception, students held all parameters constant except for the shift of the graph. Because the measure of control for these conceptions only required that the plotted graph go through some of the previously plotted points, the amplitude and shift conception sometimes contradicted one another. For example, ignoring the shift, students used the scanning tool to determine a coefficient of the cosine function that would make the graph stretch as high and as low as it needed to go. However, once they accounted for a vertical shift, they had to revisit the amplitude of the function to re-account for the vertical stretch. Even when students scanned to determine the parameters of the cosine function in a spontaneous way, these experiments supported students in recognizing a conflict between the amplitude and shift conceptions, which may have provoked them to develop a more viable conception.

x-increase-by tool. One Etoys tool that was actually problematic for students as they moved laterally between the amplitude, shift, and period conceptions of sine and cosine—the “*x-increase-by*” tool for building the script to plot their functions. The *x-increase-by* tool is an Etoys tool that allows the use to indicate how much the *x*-coordinate of the plotter should

increase by in a script. Since the Etoys script actually plotted a series of discrete points, rather than a continuous function, the x -increase-by tool was used to tell the script how much the x -coordinate of the plotter should increase before making a new stamp. The purpose of the x -increase-by tool was not obvious to students. When students assigned a very large value to “ x -increase-by,” they plotted very few points spaced very far apart. When students assigned a small value to “ x -increase-by,” they got many points very close together, to look like a continuous graph. The control structure of the amplitude, shift, and period conceptions of sine and cosine required that the graph overlap at least some of the discrete, plotted points. Students found that they could satisfy this control by increasing the value of “ x -increase-by” so that their scripts only plotted a few points, which approximately overlapped the discrete points they had previously plotted. In this way, students were able to disguise the errors in the construction of their functions by only plotting a small subset of the graph. Students’ appropriation of the x -increase-by tool actually worked against their progression towards a more viable conception of sine and cosine. Students appropriated the tool into their work in a way that allowed them to disguise the errors in the function they had constructed. To reach a more viable conception of sine and cosine, students needed to overcome this use of the x -increase-by tool.

Discussion

Three major themes emerged from the results of the analysis. First, students moved forward, laterally, and backward between conceptions of sine and cosine. Second, students’ movements in conceptions were in many ways provoked by their use of the tools of Etoys. Finally, as students’ worked on the Ferris wheel problem, their limited initial conceptions of sine and cosine indicated that this problem may have constituted a distinct *sphere of practice* for

students compared to their prior knowledge. For this reason, students did not necessarily transfer what they had previously learned about sine and cosine functions to their work on this problem.

Moving Among Conceptions of Sine and Cosine

An important implication of the findings of this study is how this framework can help to trace students' conceptions about sine and cosine functions over time. It is also important to consider how we might examine learning from students' actions over the course of the 2-day class period. In the first phase of their work, students had to translate from a circular representation of the Ferris wheel to a representation of height off the ground as a function of time. Similar to research that has found that students struggle to translate between a circle model and a functions based model of sine and cosine (Bressoud, 2010; Weber, 2005), many of the students in this study began with a conception that their height off the ground would be represented with a circle. For students who began with a circle conception, cues from the teacher pushed them to pay more attention to the meaning of the x -axis and y -axis, which pushed them towards an ordered-pairs conception of sine and cosine functions.

Once students had plotted points to represent their height off the ground at a particular time, students displayed five different conceptions of sine and cosine functions as they worked to plot a continuous function that would run through the discrete points. My observations about students' forward movements are based on the viability of students' conceptions for solving the Ferris wheel problem. The five conceptions were organized into three vertical levels, according to how viable each conception was for solving the problem. The most basic level, the symbolic conception, drew the most on students' prior knowledge of sine and cosine functions. At the next level, students' prior knowledge did not seem relevant for their work, but their use of Etoys became especially prominent. Experimenting with Etoys allowed students to move laterally

back and forth between different conceptions of sine and cosine before they moved towards a composition conception.

The conceptions framework is based on a constructivist assumption that students' conceptions exist because they are viable in a certain context and at a certain moment in time. According to the proponents of this framework (e.g., Balacheff, 2013; Balacheff & Gaudin, 2002), talking about student thinking in terms of *misconceptions* is misleading, because it does not take into account the details or idiosyncrasies of the many different ways of thinking about a single concept. Still, conceptions can be examined in terms of how viable they are for solving a particular problem. In the context of the Ferris wheel problem, moving from a symbolic conception of sine and cosine functions to a composition conception indicated a move from a conception that was not viable for solving the problem towards the most viable conception for solving the problem.

In the cK ϕ framework, learning is defined as a process that allows an individual to move from one conception to another (Balacheff & Gaudin, 2003, p. 15). Imposing a hierarchy on the conceptions of sine and cosine in the context of the Ferris wheel problem gives an additional requirement that learning should enable an individual to move from a less viable conception of sine and cosine towards a more viable conception. One could infer that students who moved from the symbolic conception of sine and cosine functions towards the composition conception learned something over the course of their work on the problem. The lateral movement between conceptions of sine and cosine functions can be thought of as perturbations of students' conceptions. A mathematical problem can provide a context for students to come to understand a topic not only with but also *against* their previous conceptions of that topic (Herbst, 2005). The Ferris wheel problem provided a context for students to come to know about sine and cosine

functions against their previous symbolic conceptions, and then against each of the amplitude, shift, and period conceptions. Through their work on the problem, students learned about sine and cosine functions. Specifically, they moved from less viable towards more viable conceptions for solving the problem.

Students' Instrumented Activity With Etoys

Given the evidence that students learned about sine and cosine functions through their work on the Ferris wheel problem, there is a question about how much students' learning was instrumented through their use of Etoys. One might expect, especially considering that students had some prior knowledge of sine and cosine functions, that students' inexperience with Etoys was their main barrier towards solving the Ferris wheel problem. In such a scenario, students may be able to develop a mathematical solution to the problem fairly quickly, but they would need to learn how to use the tools in Etoys for the purposes of writing a script. It is important to consider whether this hypothetical scenario was what actually played out in students' work. If students spent two days learning how to use Etoys, apart from their mathematical activity, then their learning about sine and cosine functions would not actually be instrumented through their use of the Etoys tools. This would suggest that, had students worked on the problem with paper and pencil, they could have bypassed the less viable conceptions and immediately displayed composition conceptions of sine and cosine in their work.

There was evidence, however, that Etoys was not the cause of students' relatively limited initial conceptions of sine and cosine. Specifically, students had paper and pencil, and they could have used those materials to solve the problem, though most did not. The worksheet containing the Ferris wheel problem included questions for students make written predictions about what function they would use to model the scenario. Students were encouraged to use the

space provided on the paper to come up with the function they needed. Ms. Alexander even suggested to multiple pairs of students that they “write out a cosine equation” on the paper in front of them. In short, had Etoys been the sole cause of students’ limited conceptions of sine and cosine, students could have abandoned their use of Etoys and solved the problem with paper and pencil, as was the case with one pair of students. The fact that most groups of students did not use paper and pencil suggests that their conceptions of sine and cosine would not have been viable for solving the problem, even if they were working in a paper and pencil environment.

Students’ use of the tools in Etoys was integral to their conceptions of sine and cosine, which is especially apparent when considering the different components of the cK ϕ framework. Students made extensive use of the Etoys tools as part of the operations that contributed to their conceptions. When students attempted to use the drag-and-drop tool within a symbolic conception, their operations failed. Rather than abandoning their use of that tool, students began to appropriate the drag-and-drop tool into their mathematical work. In other words, students began to use the drag-and-drop tool for the “purpose it embodied” (Leontiev, 1981), namely, to construct a function in which the mathematical relationships among components of the function were explicit. Students’ appropriation of the drag-and-drop tool connected their mathematical work with the work of writing a script in Etoys. This was evidence that students’ learning was in fact instrumented through their use of the drag-and-drop tool. Students’ thinking about sine and cosine functions occurred through their use of the drag-and-drop tool.

In addition to the drag-and-drop tool, students used the scanning tool in ways that provoked them to move between amplitude, shift, and period conceptions of sine and cosine functions. When students appropriated the scanning tool in a deliberate way, they used the tool to test and correct their conjectures about the appropriate parameters of the sine or cosine

function. Even when students were not deliberate with the scanning tool, it still supported the activity of making connections between the symbolic representation in the Etoys syntax and the graphical output. For example, students could notice that increasing the coefficient of a sine function would increasingly stretch the output in a vertical direction. Connecting these different representations allowed students to determine whether they had been able to satisfy the control structures of different conceptions. Similarly to the drag-and-drop tool, students used the scanning tool in ways that were integral to their mathematical work.

The case of Zach and Jalisa provided the single counterexample to my suggestion that students' learning was instrumented through their use of the tools in Etoys. Zach and Jalisa were one of the two pairs of students who immediately showed evidence of the composition conception of sine and cosine in their work. Zach and Jalisa did not construct the function to represent the Ferris wheel situation with Etoys, but instead they wrote the function down with pencil and paper. Zach and Jalisa provide an example of students who did not appropriate the tools of Etoys as part of their mathematical work. They certainly used the tools, because after they formulated their mathematical solution, the pair spent much of the class period using the drag-and-drop tools to try to make a script. However, Zach and Jalisa's use of Etoys was not integral to any of the operations, representations, or control structures that they used to solve the two parts of the Ferris wheel task. Once Zach and Jalisa achieved a solution to the problem, working with Etoys became a secondary task for them to represent their solution with a graph through the points they had plotted. Their mathematical learning was not instrumented in the ways that other students' learning was.

The evidence from this study suggests that, overall, students' learning about how to use the Etoys software was not something that occurred separately from their conceptions of sine and

cosine. Instead, students learned about sine and cosine functions as they learned how to construct functions with Etoys. Not only did students' learning about Etoys happen at the same time as their learning about sine and cosine, but actually students' learning about Etoys *supported* their learning about sine and cosine. Students used the Etoys tools for exploring how the different components of a cosine function would affect the shape of a plotted graph. Students' mathematical learning became instrumented as they appropriated the tools of Etoys for the purpose of learning about sine and cosine functions. Students used the drag-and-drop tools to examine different compositions of functions. Students used the scanning tool in a way that allowed them to explore the parameters of the cosine functions. The tools that students used became instruments as they appropriated those tools in a way that helped them learn about sine and cosine functions.

Spheres of Practice of Sine and Cosine Functions

The evidence of very limited initial conceptions of sine and cosine functions was a somewhat surprising finding in this study, given that the Ferris wheel problem occurred at the end of a unit on trigonometric functions. Returning to the notion of different *spheres of practice* (Bourdieu, 1990) offers some insight on why students' conceptions may have been initially very limited in their work on the problem. Spheres of practice refer to the mutually exclusive—from the perspective of the student—domains of validity in which conceptions may occur. Different conceptions, which may otherwise contradict each other, do not contradict because they occur within different spheres of practice. Spheres of practice may be separated by time; for example conceptions of sine and cosine in Algebra 2 may not conflict with conceptions of sine and cosine from Geometry because they occur during different school years. In addition to that, students

may not see their conceptions from Geometry as relevant to what they study in Algebra, because these two contexts are distinct.

Apart from time, spheres of practice may be separated by context. It seemed that working on the Ferris wheel problem constituted an entirely new sphere of practice for students than their prior knowledge of sine and cosine functions, even though students had studied these functions in just the days and weeks prior. Once students had plotted their discrete points, their task was tantamount to constructing a sinusoidal function to represent a Ferris wheel ride that had maximum and minimum heights of 135 and 5, respectively, and which took 30 minutes to complete one revolution. To an outside observer, this task is exactly equivalent to writing a sinusoidal function whose maximum and minimum are 135 and 5, respectively, which has a period of 30. Students had solved similar problems in the past, and students quickly solved a similar problem during a warm-up on day 2 of the lesson.¹⁰ However, students did not seem to bring their experiences from these strictly symbolic problems to the Ferris wheel problem.

There are multiple explanations for why the Ferris wheel problem may have constituted a distinct sphere of practice from the perspective of the students in the study. First, the Ferris wheel problem was the first contextual problem that students worked on with sine and cosine functions. Prior to that, students had worked with sine and cosine functions symbolically and graphically, but they had not connected sine and cosine functions to any real-world context. Second, the Ferris wheel problem was students' first experience using Etoys, and it was a rare experience of going to math class in the computer lab. It is possible that working on the computer, and using a programming environment, was such a novel experience for students that

¹⁰ Day 2 of the lesson began with a warm-up where students had to write “a cosine equation with a minimum of 10 and a maximum of 30,” and “a cosine equation with a period of 15.” See Appendix C.

they did not expect their prior knowledge to apply. Given my previous justifications for why students' conceptions were not entirely reducible to their inexperience with Etoys, I would expect that the novel experience of using Etoys was not the primary factor in contributing to the distinct sphere of practice. But I expect that both factors —working on a contextual problem over two days and using a computer-programming environment—worked together to make the Ferris wheel problem a new sphere of practice in which students would invoke conceptions of sine and cosine.

Students' symbolic conceptions of sine and cosine were one way that students attempted to translate their work on previous sine and cosine problems to their work on the Ferris wheel problem. Students had some prior knowledge of what a symbolic representation of sine or cosine should look like, and they tried to use this knowledge towards their work on the Ferris wheel problem. The difficulty with the symbolic conception was that it did not take into account the meaning of the various components of a sine or cosine function, or the relationship between the symbolic and graphical representations of the function. The two pairs of students who quickly achieved a composition conception of sine and cosine did seem to bring their prior knowledge of sine and cosine functions to their work on the problem, and those students established a correct solution to the problem relatively quickly.

It has been documented that there is often disconnect between different contexts of doing mathematics, specifically with the distinction between in-school and out-of-school mathematics (Abreu, 1995; González, N., Andrade, Civil, & Moll, 2001; Lave, 1988; Rogoff, 1990; Saxe, 1991)¹¹. Even within school mathematics, when students encounter a familiar concept in a novel

¹¹ In a study comparing school mathematics to workplace practices, Roth (2012) has begun to make explicit connections between in-school treatment of trigonometric ratios and the use of those ratios in electrical conduit bending in the workplace. Although this was a study of college

setting those conceptions do not necessarily accommodate the new setting (e.g., González, G. et al., under review; Herbst, 2005; Martínez-Planell et al., 2012; Miyakawa, 2004). Sine and cosine functions make up a conceptually rich part of the mathematics curriculum, specifically because they are relevant in a variety of mathematical and real world contexts. However, this study points out a disconnect between these different contexts, even when they appear the same from the perspective of an observer. To support students understanding of sine and cosine functions, it is important to identify ways to build connections among what are, for students, distinct contexts. Ultimately, by building these connections, students should be able to develop more robust understandings that translate to new situations.

Conclusion

I have described what conceptions of sine and cosine functions students displayed over two days of work on the Ferris wheel problem. I also examined how students shifted among the different conceptions of sine and cosine, and how students' use of Etoys may have provoked changes in conceptions. Students' conceptions fell into a hierarchy in terms of their viability for solving the Ferris wheel problem. By moving from less viable to more viable conceptions of sine and cosine functions, students showed evidence of learning through their work on the problem.

Students' use of Etoys gave them a way to experiment with the parameters of a sine or cosine function and examine the relationship between syntactical and graphical representations of the function. Using Etoys forced students beyond a symbolic conception of sine and cosine, because with Etoys students could not satisfy the control structure for the symbolic conception. In addition, students' use of Etoys caused them to shift between amplitude, shift, and period

students who were also apprentice electricians, it indicates a step to make the connections between different spheres of mathematical practice explicit.

conceptions. In both of these ways, Etoys supported students' learning about sine and cosine functions. Learning how to use the features of Etoys was an integral process of learning about the functions.

There are two potential areas of future research that have emerged from this study. First, an important question is how students' use of a programming environment may impact their learning about trigonometric functions and other families of functions over time. There may be a contrast between situations where students are already proficient with a programming environment versus situations where students learn about a programming environment at the same time as learning about a mathematics concept. This study gave specific attention to the latter case. Students' learning about sine and cosine was intertwined with their learning about Etoys. Moreover, learning how to use Etoys actually helped students learn about sine and cosine functions, as in the case of using the Etoys syntax to move beyond a symbolic conception. Students who were already proficient with the tools of a programming environment would likely use those tools differently, and as a result their learning of mathematics would be different. To better understand how students' mathematical activity is instrumented through their use of programming tools, a promising area of research would be to examine how students' use of those tools evolves over time.

Finally, it is important to keep in mind that students' work on the Ferris wheel problem came at the end of a unit on trigonometric functions. Students in the study had previously solved problems about sine and cosine functions, which from the perspective of an observer were isomorphic to the Ferris wheel problem. Students, for the most part, did not bring their prior knowledge to their work on the Ferris wheel problem. This indicated that, for students, the Ferris wheel problem constituted a distinct sphere of practice from what they had previously

experienced with sine and cosine functions. Future research can examine how to support students' learning about trigonometric functions in a way that makes an explicit connection between these different spheres of practice. These connections may be forged through the design of tasks, the tools that students use, or the actions a teacher takes to use students' prior knowledge. Helping students to build bridges between distinct domains of validity can support students in seeing connections in mathematical ideas between different types of problems and contexts.

CHAPTER 3:

STUDENT LEARNING ABOUT SINE AND COSINE FUNCTIONS THROUGH WORK ON A CONTEXTUAL PROBLEM WITH ETOYS

Trigonometric functions, including sine and cosine, present an interesting challenge regarding the study of function in high school. In many American curricula, students study the concept of sine and cosine in multiple courses, and with emphasis on different aspects of this concept. High school students most often encounter trigonometric functions first in the context of right triangle trigonometry, which is often situated in the Geometry course (e.g., Bass, Bellman, Bragg, Charles, Davidson, Handlin, & Johnson, 2004; Burger et al., 2007; Dietiker et al. 2007). At this time, sine and cosine are ratios computed in right triangles. When students study sine and cosine in Geometry, sine and cosine are not actually treated as functions. Later, when students take Algebra 2 and Pre-Calculus, they use the unit circle to extend the domain of trigonometric functions to include all real numbers (e.g., Day, Hayek, Casey, & Marks, 2004; Dietiker et al., 2006; Shultz, Ellis, Hollowell, & Kennedy, 2007). In typical curricula, the unit circle is introduced to support the transition from sine and cosine as computed in triangles and towards understanding sine and cosine as functions that can be represented as graphs on the Cartesian plane. Research suggests that this transition creates a unique challenge for students studying trigonometric functions that does not surface with other families of functions (Thompson, 2008; Weber, 2005). It may be especially difficult for students who have previously encountered sine and cosine as ratios in right triangles to consider sine and cosine as functions of an independent variable in later grades.

The representations through which students study trigonometric functions are critical for how students understand these functions (Blackett & Tall, 1991; Breidenbach et al., 1992;

Bressoud, 2010; Moore, 2013; Thompson, 2008; Thompson et al., 2007; Weber, 2005).

Translating between right triangles, the unit circle, and the Cartesian plane may contribute to the challenge for students to develop a strong conceptual understanding of sine and cosine as functions. For example, having studied sine and cosine as ratios in right triangles, students may see these as functions that take triangles as inputs (Thompson, 2008). There is no clear consensus about which representations of trigonometric functions promote the strongest conceptual understanding of these functions in students. However, it seems that the process of constructing, and making connections between, different representations of trigonometric functions is crucial for students' sense making (Kendal & Stacey, 1997; Moore, 2013; Weber, 2005). To develop competency working with sine and cosine functions, it is important for students to make connections between symbolic, visual, and graphical representations of these functions.

Typical textbooks contain many examples for representing periodic real world contexts with sine and cosine functions. There are a multitude of real world contexts that follow sinusoidal patterns, including the motion of a Ferris wheel, the rise and fall of tides, the hours of daylight in a day, and the change in temperature over the course of a year. These contexts constitute canonical problems that can be found, with slight variations, in a variety of Algebra 2 textbooks (e.g., Day et al., 2004; Dietiker et al., 2006; Larson, Boswell, Kanold, & Stiff, 2004). These problems are aligned with the *Common Core Standards for Mathematics*, which call for students to “model periodic phenomena with trigonometric functions” (NGAC, 2010, p. 71). One could expect that students would more fully understand the connections among different representations of sine and cosine through working on problems situated in real world contexts, which require translations between different representations of that context.

The importance of representations for studying periodic, real-world situations presents an opportunity to examine students' thinking about sine and cosine functions from working on a problem of this nature. In this study, I have sought to examine changes in student thinking after students worked for two days on an open-ended problem that required them to make connections between multiple representations of the problem. I explored whether and how students' thinking about sine and cosine functions improved after using a computer-programming environment to represent a context about riding a Ferris wheel using sine and cosine functions.

Research Questions

This study is guided by RQ2: *What evidence did students show of learning about sine and cosine functions through their work with Etoys?* This research question can be further subdivided into three sub-questions:

1. How did students' performance on problems related to sine and cosine functions change from a pre-test to a post-test?
2. How did students who were low-achieving versus high-achieving on the pre-test compare in their change in performance from the pre-test to the post-test?
3. How did students' strategies on individual tasks during a post-lesson interview compare with the strategies that they used on similar tasks during work in pairs?

Taken together, these questions address the topic of students' learning from a large-scale perspective—considering the aggregation of students' scores on pre- and post-tests, and from a small-scale perspective—inquiring more deeply into the conceptions that surfaced in students' work in a one-on-one, task-based interview.

Theoretical Framework

The guiding framework of this study includes two components. First, I use a *constructivist* perspective of learning to make sense of students' conceptions of sine and cosine functions. Second, I use *instrumented activity* as a way of understanding how students' work on a problem with Etoys may have shaped their understanding of sine and cosine functions.

Constructivism and Students' Conceptions

The tradition of student conceptions research in mathematics education has emerged from the foundations of the constructivist paradigm (Confrey, 1990). Namely, research on student conceptions grew out of Piaget's micro-analysis of how students developed specific concepts. Broadly speaking, students' *conceptions* refer to the categories of students' beliefs, thinking, and explanations. Largely following the work of Piaget, research has given attention to the processes by which students develop their knowledge of mathematics. An early example of this perspective was provided by Erlwanger's (1973) study of Benny, a student who performed relatively well on standardized measures of achievement. After a series of interviews with the student, Erlwanger discovered many idiosyncrasies in Benny's thinking that were not consistent with commonly accepted mathematical thought. Studies such as Erlwanger's, giving evidence of how students explain and justify mathematical ideas, further solidified the case for research in mathematics education giving more attention to student thinking in addition to measurable outcomes.

A large number of more recent studies have given empirical evidence that individual students develop different ideas, and those ideas are often different from canonical ways of thinking about mathematics (Confrey, 1990; Hiebert & Carpenter, 1992). Moreover, the intricacies of students' thinking are not always immediately obvious from an observer's

perspective. Elementary school students are quick to acknowledge that a dime is worth 10 cents; but they often struggle to subtract, for example, 6 cents from a dime, because they treat a dime a single unit (Chandler & Kamii, 2009). When students begin to learn fractions, they can think of a fraction such as $\frac{3}{5}$ as either a part of a whole, or as three iterations of a unit of $\frac{1}{5}$ (Norton & Wilkins, 2012; Steffe, 2003). This difference has implications for the operations that students can perform with fractions. In later grades, students may seem to have solid understandings of reflective symmetry in tasks of construction, but then they do not translate the same knowledge to tasks of proving (Miyakawa, 2004). This small collection of studies illustrates a much larger phenomenon that has emerged from the tradition of research into students' conceptions in mathematics. Namely, students have ways of thinking about mathematical ideas that are not always consistent and not always what an outside observer would expect or immediately observe. This makes salient the point that research on students' mathematical thinking and learning should give a detailed examination of students' work to uncover the idiosyncrasies in students' thinking.

The cK ϕ framework for understanding students' conceptions (Balacheff & Gaudin, 2002, 2003) contributes to research on students' conceptions by providing an operational definition of what a conception is. The cK ϕ acronym refers to the constructs of "conception, knowing, concept." While a *concept* is an idea that is commonly accepted by the mathematical community, a students' *conception* refers to the students' beliefs, theories, and explanations of a particular concept. The cK ϕ framework operationalizes the construct of conception by establishing a link between students' behaviors and the thinking those behaviors suggest. From this perspective, although thinking cannot be reduced to behaviors, behaviors are valuable in that

they give insight into students' thinking (Balacheff & Gaudin, 2002). A *conception* is defined as a quadruplet $C = (P_C, R_C, L_C, \Sigma_C)$:

- P_C is the set of problems or tasks in which the conception is used,
- R_C is the set of operations that a student may use in completing the problems in that set,
- L_C is the system of representations in which the problem is posed and their solutions are expressed,
- Σ_C is the control structure, or the way of knowing whether the solutions expressed in the set of problems is correct.

The first three components of the cK ϕ framework were earlier identified by Vergnaud (1982, 1983, 1988, 1998, 2009) to characterize a concept. At the foundation of a conception is a mathematical problem, or set of problems, to be solved. Operations are the “tools for action” (Balacheff & Gaudin, 2002, p. 7). The system of representations can include, for example, algebraic language, graphical representations, sketches, or computer interfaces. Representations give account of the problem and allow the student to perform operations. Finally, the control structure allows the student to verify, from his or her own point of view, that the actions performed are appropriate, the solution is correct, or that a problem is solved. A control structure can be thought of as what motivates a student's action (Balacheff, 2013). Whether or not a conception is actually viable for solving a particular problem, the control structure serves to motivate and check the operations of the conception.

In this study, the cK ϕ framework provides a connection between students' observable actions and the mathematical thinking those actions could reflect. The four components of a conception have allowed me to identify how students differed in their thinking. For example, two students may use the same representations to solve a problem but perform different

operations, which reflects differences in how students think about a concept. In addition, by identifying the control structure of a conception, I have been able to understand why students' thinking may be inconsistent with standard mathematical thought. The cK ϵ framework builds from the assumption that even though a conception may not be true according to an expert's knowledge, there is likely some legitimacy to that conception. The control structure of a conception reveals the motivations behind the operations one would use. In that way the control structure explains the legitimacy of a conception from the student's point of view. The cK ϵ framework privileges the idea that student thinking most often stems from prior knowledge that was viable in some other context. Using this framework, I have been able to compare student thinking along the four components defining a conception, and I have been able to infer how students' prior knowledge and experiences contributed to their conceptions of sine and cosine.

Instrumented Activity

The theory of instrumented activity is a way to explain how students' thinking about mathematical ideas is shaped by the tools they use. This theory follows from the assumption that the nature of all human activity depends on the ways that individuals use different tools (Verillón & Rabardel, 1995). Thinking specifically of technology tools, Verillón and Rabardel argued that students' use of technological tools changes how they think about the ideas they encounter through the use of those tools. In mathematics education research, students' use of technology tools is integral to how students think about mathematical ideas (Heid, Blume, Flanagan, Iseri, & Kerr, 1999; Hollebrands, 2003; Laborde, 2001; Noss & Hoyles, 1996). In light of the conceptions framework, it is easy to see that the representations available to students for engaging with a particular concept would be different when using technology tools versus not. The operations students would perform would likely be different, in addition to the control

structures for verifying whether those operations were appropriate. The nature of the problems that students would work on may even be different when using technology tools versus not. It is reasonable to expect, therefore, that students' use of technology tools could be integral to their mathematical conceptions.

It is important to make a distinction between artifacts, tools, and instruments to explain how students would use technology tools for learning mathematics. An *artifact* refers to a material object, to which an individual has access (Leontiev, 1981). A straight edge is an artifact that may be present in various forms in mathematics classrooms. What distinguishes a *tool* from an artifact is not only its physical properties, but also the way the tool is used for a specific purpose, which is elaborated socially (Leontiev, 1981). For example, a ruler is a tool that has been designed in a specific way for the study of mathematics, to be used as a straight edge and also as a device for measuring. While an artifact could be any material object available in an environment, a tool is an object that has been designed for some purpose.

A computer-programming environment is an example of a *microworld* that contains multiple tools. A microworld is an environment, based in a computer or another medium, in which the objects and relationships of a particular domain are made concrete (Edwards, 1991; Hoyles & Noss, 1987; Papert, 1980). In this case, the domain is a particular domain of mathematics, meaning that a microworld allows users to interact with concrete representations of mathematical objects and relationships. A computer-programming environment is a microworld in which the user uses a programming language, and therefore maintains symbolic control of the activities in the environment (Healy & Hoyles, 2001). This feature distinguishes a computer-

programming environment from other microworlds, for example a dynamic geometry environment in which a user interacts by directly manipulating objects on the screen¹².

For the purpose of research it is revealing to examine the different tools that students may use within a programming environment, particularly since students are likely to interact with only a subset of the available tools. Prior research has identified different tools that are offered to students through technology environments, particularly in the case of dynamic geometry environments. For example, researchers have examined the different ways that students use the *dragging* tool and *measuring* tool of Geometer's Sketchpad and similar dynamic geometry environments (e.g., Arzarello et al., 2002; Hollebrands, 2007; Laborde, 2001; Olivero & Robutti, 2007). Research in the use of computer-programming environments in mathematics education has examined how these environments allow students to make connections between symbolic and visual representations of mathematical ideas (e.g., Clements & Battista, 1989, 1990; Edwards, 1991, 1997; Hoyles & Healy, 1997; Hoyles & Noss, 1992). In addition, this body of research has identified the advantages of allowing students to maintain symbolic control of their work through a programming language (di Sessa, 2000; Hoyles & Sutherland, 1989; Hoyles & Noss, 1992). Research on the advantages of programming environments may be expanded by enumerating the specific tools available through various programming environments in a way analogous to the various tools identified in dynamic geometry environments. In another study I have identified three of the tools available within the Etoys programming environment that students used to work on a problem about sine and cosine functions—the *drag-and-drop* tool, the *scanning* tool, and the *x-increase-by* tool (see Chapter 2). The important point here is that

¹² Developments in the design of Dynamic Geometry Environments have increased users' capabilities to directly input commands. GeoGebra (Hohenwater, 2001), for example, contains an input bar through which users can enter syntax.

students used a subset of the tools available within the Etoys environment in ways that shaped their mathematical understanding.

Verillón and Rabardel (1995) took a further step to distinguish artifacts and tools from *instruments* of human activity. An instrument is an object that has been appropriated by an individual for a specific purpose.

But it is important to stress the difference between the two concepts: the artifact, as a manmade material object, and the instrument, as a psychological construct. The point is that no instrument exists in itself. A machine or a technical system does immediately constitute a tool for the subject. Even explicitly constructed as a tool, it is not, as such, an instrument for the subject. It becomes so when the subject has been able to appropriate it for himself...and, in this respect, has integrated it with his activity (pp. 84-85).

Verillón and Rabardel distinguished between the artifact and tool—as physical objects, and the instrument—as a psychological construct. Tools, although designed for specific purposes, do not become instruments for an individual until the individual can appropriate those tools into his or her activity in a purposeful way. Individuals appropriate a tool as they engage the tool for a purpose according to how it has been designed (Vygotsky, 1930). When individuals appropriate tools, integrating them into their activity for some specific purposes, those tools become instruments.

When students engage in instrumented activity in mathematics, they use the tools provided by an environment to make sense of mathematical ideas. In my previous study on students' use of the tools in the Etoys environment, I found that different students appropriated the various tools in different ways (see Chapter 2). For example, some students used the scanning tool in a very deliberate way, to test a conjecture about what the coefficient of a

function would be and to make adjustments based on their observations. Other students used the scanning tool in a seemingly random way, scrolling through numbers and observing the output of the script, without making reasoned conjectures or responding to the connections between the input and the output of the syntax. As students appropriated the tools of Etoys for their work on the Ferris wheel problem their activity became instrumented, in that their use of the instruments became integral to their thinking about sine and cosine functions.

The Instrumented Activity Situations [IAS] model (Verillón & Rabardel, 1995) offers a view of how students' interactions with the concept of sine and cosine functions are instrumented through their use of Etoys (Figure 3.1). In a situation of instrumented activity there are three key elements: the subject, the object of the subject's action, and the instrument. In the setting of this study, the subject is a student. The object of the student's action is the mathematical concept of sine and cosine functions. The instruments emerge when students appropriate the tools from Etoys for their work, as in the example of the scanning tool. Each component of the triad interacts with the other two. Most importantly in the IAS model, students interact with the concept of sine and cosine functions through their work with the tools of Etoys. Students' understanding of the mathematical ideas are shaped by what they do with the technology tools.

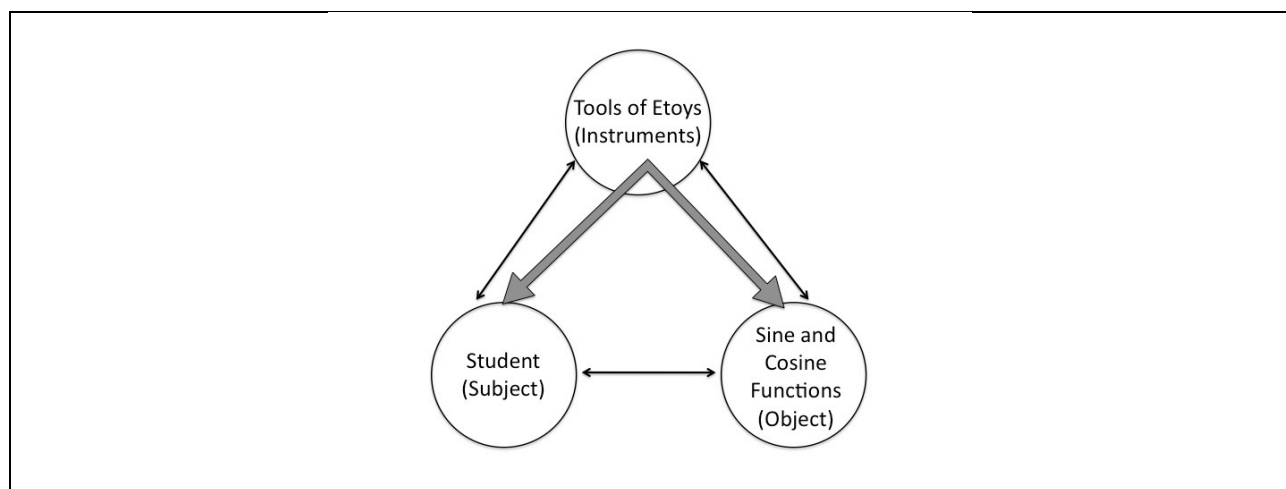


Figure 3.1. The Instrumented Activity Situations [IAS] model (Verillón & Rabardel, 1995).

Instrumented activity does not require the introduction of a technology environment. Even in a non-technology based lesson, students' interactions with sine and cosine functions would depend in some ways on their use of tools, for example pencils and graphs. The importance of the IAS model in this study is that the tools of Etoys were new tools in students' work, included in addition to the more traditional tools such as written and symbolic language, paper, and pencil. For this reason, one could expect that students' use of the Etoys tools would create new ways of thinking about sine and cosine functions that may not have been otherwise available.

Data and Methods

Subjects

The participants of this study were Algebra 2 students at Grove High School¹³. Grove High School is a school of approximately 1000 students, where 30% of students qualify for free or reduced lunch. The student population at Grove High School is around 60% White and 30% Latina/o. The teacher, Ms. Alexander, taught three sections of regular (i.e., non-honors) Algebra 2. All of the students in Ms. Alexander's sections participated in the Etoys lesson, although not all students participated in the study. Twenty-eight students participated in the pre-test and post-test portion of the study. Of those students, 23 students (10 pairs and one group of three) agreed to be video and audio-taped during the Etoys lesson. Finally, 12 students participated in a one-on-one post-lesson interview. For the interview portion of the study, I included all students who elected to participate and who were available during a free period or before or after school.

Table 3.1 lists students who participated in each phase of the study.

¹³ I use pseudonyms for all people and places.

Table 3.1
Students who Participated in Each Phase of the Etoys Study

Participated Only in Pre-Test and Post-Test Data Collection	Participated in Video-Taping of Etoys Lesson		
Mariam	Carson	Gia*	Hannah
Trent	Abbey	Courtney*	Dayana
Jessa			
Eva	Bailey	Mike*	Mitchell*
Bailey	Aubrey	Jessa	Reese
Erica			
	Cara*	Tori*	Shane*
	Maggie*	Sean	Maya
	Jalisa	Lucas*	
	Zach*	Elizabeth*	
		Andy*	

* Indicates students who also participated in post-lesson interview.

Procedure

Ms. Alexander implemented the Etoys lesson over two consecutive days during the spring semester of the school year. The lesson came at the end of a larger unit on sine and cosine functions. Ms. Alexander had a classroom textbook that she used for reference (Day et al., 2004), but she did not follow the order of the textbook. Students had previously studied the amplitude, period, and vertical and horizontal shifts of sine and cosine functions. They had worked on graphing different sine and cosine functions and writing functions to satisfy certain properties. Ms. Alexander had saved the real world applications until the end of the unit, so students in Ms. Alexander's class had not yet used sine and cosine functions to represent real world phenomena. I designed the Etoys lesson, in consultation with my advisor and other experts in the use of Etoys for educational purposes (González & Lundsgaard, personal communication, 2013; Pitt, personal communication, 2013). The purpose of the Etoys lesson was to address standard F-TF.5 from the *Common Core State Standards for Mathematics*:

“Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline” (NGAC, 2010, p. 71).

Over the two-day Etoys lesson, students worked on a problem called the Ferris wheel problem. On the Ferris wheel problem, students had to imagine that they were riding a famous Ferris wheel (see Figure 3.2). Given the diameter of the Ferris wheel and the time it takes to make one complete revolution, students had to write a function that would represent their height off the ground as a function of time while riding the Ferris wheel. For a complete solution to the Ferris wheel problem, see Appendix B.

One of the most famous Ferris wheels in the world is the London Eye in London, England. Assume that the London Eye has a diameter of 130 meters, and the lowest point on the Ferris wheel is 5 meters above the Thames River. It takes 30 minutes to make one complete revolution. You and your partner are going to ride the Ferris wheel. You get on the Ferris wheel at the very lowest point.

Figure 3.2. The Ferris wheel context.

On the first day of the Etoys lesson, Ms. Alexander launched the problem by showing pictures of the London Eye Ferris wheel and giving an overview of her expectations for students' work on the problem. Ms. Alexander told her students that she would be grading the assignment as a “rich task,” which meant that students were expected to be talking about the problem with their peers, working on the problem for the duration of the class period, and relying on their peers rather than the teacher to answer questions. Students were used to working on rich tasks in Ms. Alexander's class, and they did so around once per week or once every other week. Ms. Alexander told her students that they would not be graded on whether they got the “right answer” to the problem, but that they needed to come up with a solution to the problem. After Ms. Alexander launched the problem, students spent the duration of the class period working with their partners to solve the problem. On the second day of the lesson, students spent the first 5

minutes of class working on a warm-up problem about sine and cosine functions that was a review of prior material and not related to any real world application (see Appendix C for the warm-up problem). After the warm-up, students continued their work on the Ferris wheel problem. Ms. Alexander spent around 10 minutes at the end of class on the second day of the lesson providing closure to students' work on the Ferris wheel problem and establishing a correct solution. Students took the pre-test during the week before the Etoys lesson. I conducted the one-on-one interviews with students the day after the conclusion of the lesson, and students also took the post-test on the day after the conclusion of the Etoys lesson.

Features of the Etoys Environment

To work on the Ferris wheel problem, I provided students with an Etoys notebook, which is a pre-constructed file with different pages of resources for students to use during their work on the problem. On the first page of the notebook was a virtual representation of the Ferris wheel (see Figure 3.3). The representation intended to support students to visualize when they would be at certain heights off the ground if riding the Ferris wheel. Students could make the Ferris wheel move so that they could trace the height of one carriage on the Ferris wheel.

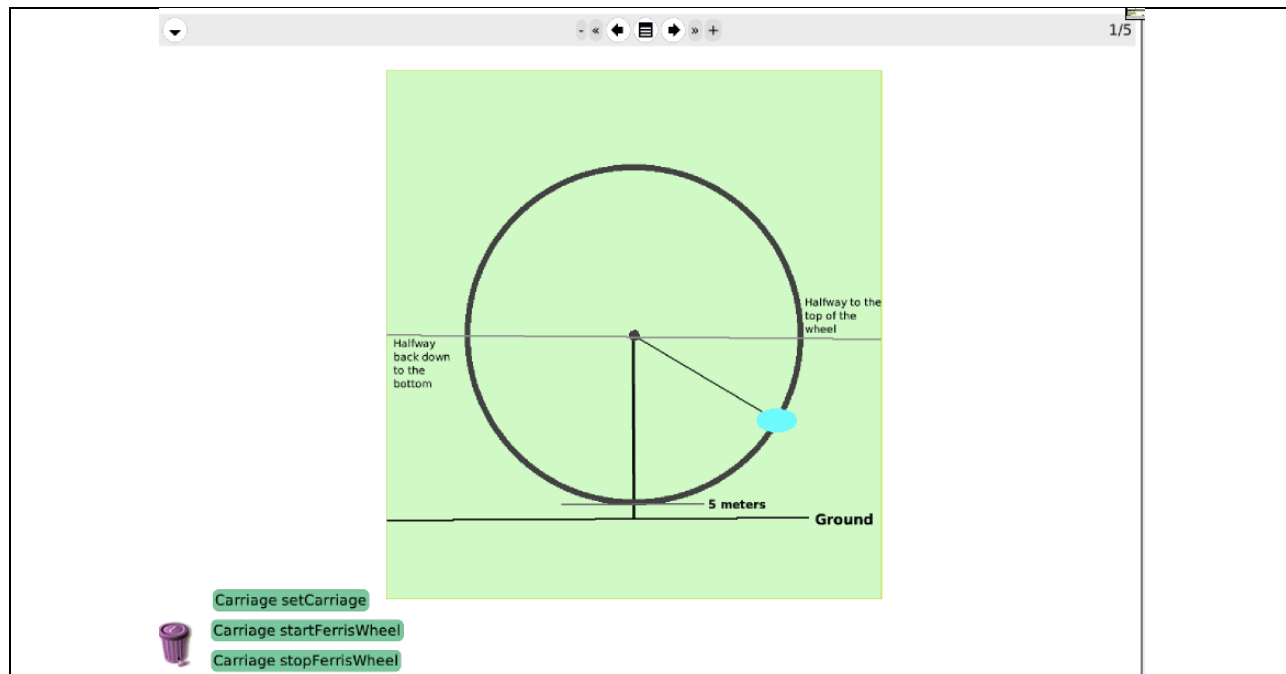


Figure 3.3. The virtual Ferris wheel representation on page 1 of the Etoys notebook.

On the second page of the notebook, students worked with an already-constructed script that would plot a quadratic function (Figure 3.4). The purpose of this page was for students to observe how to write a script in Etoys. In the script, a green dot on the page acted as the “plotter,” a programmable object. The first command in the script was to assign a y -value to the plotter based on the x -value. The second command in the script directed the plotter to “stamp,” or to make a mark on the grid. The third step in the script increased the plotter’s x -value by some fixed amount before repeating the first action. With the design of the lesson, I expected that students would create their own scripts based on the script provided on this page. The main difference in the script students would produce would be in the rule that students used to assign a y -value in the first command of the script. The rule provided in the example was a quadratic function, but students would need to use a sinusoidal function to represent their height off the ground while riding a Ferris wheel.

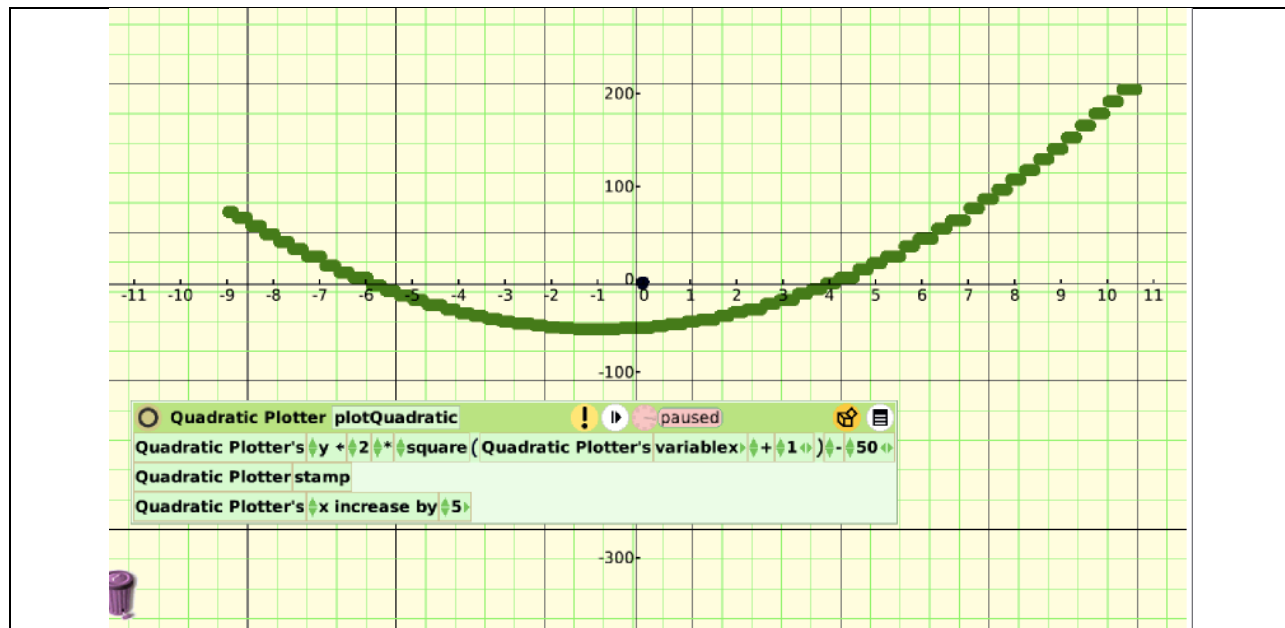


Figure 3.4. The quadratic script on page 2 of the Etoys notebook.

During her launch of the problem, which she repeated very similarly in each class period, Ms. Alexander used the script on page 2 of the Etoys notebook to give students a brief overview of how to use Etoys. Specifically, Ms. Alexander showed students that they could click on the different numbers in the script to change them, or they could use the arrows to increase or decrease numbers. Ms. Alexander also pointed out to students that the “square” tile referred to the squaring function, and that students could change the function by clicking on the tile and selecting from a drop-down list. Last, Ms. Alexander showed students that they could drag available tiles into the script to change what was currently there. The purpose of Ms. Alexander’s overview of Etoys was to illustrate for students how to use the basic features of the software. Ms. Alexander did not say anything to students about how they would use Etoys to solve the Ferris wheel problem.

Page 3 of the Etoys notebook provided students with an empty graph, a plotter, a collection of discrete points for them to drag-and-drop, and a collection of drag-and-drop menu

items to construct their script (Figure 3.5). First, students used the points to plot their height off the ground at several different moments in time. Based on the points they had plotted, students had to write a script that would make a graph running through those points. It was at this phase of the lesson when students would need to draw on their prior knowledge of sine and cosine functions to write their scripts. I designed the lesson with the intention that students would construct their functions in Etoys. However, Ms. Alexander also told students that they could write their solution on paper if they felt that they knew the solution to the problem but did not know how to use Etoys.

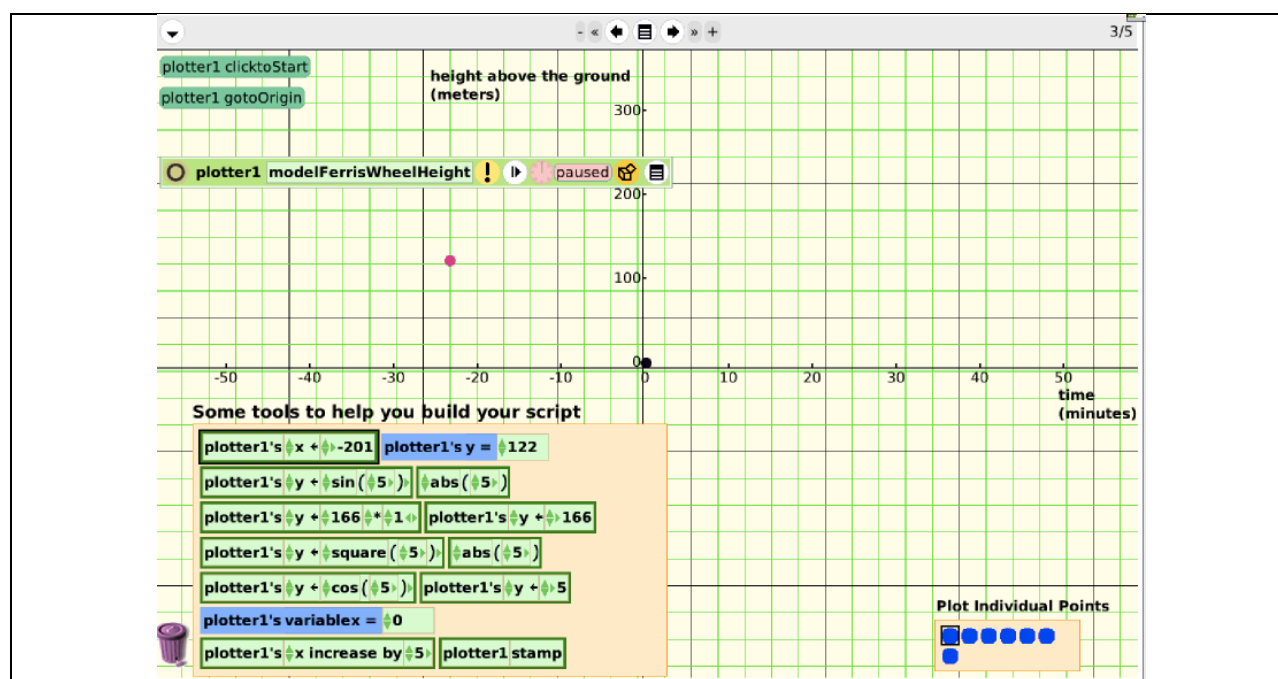


Figure 3.5. The features provided to students on page 3 of the Etoys notebook.

Pre- and Post-Tests

The pre- and post-tests contained 6 items (see Appendix D). Five of the test items came from the released items from the National Test of Educational Progress [NAEP]. The released NAEP items did not contain any items directly related to representing real world contexts with sine or cosine. I constructed a pair of items of this nature, based on typical textbook problems

(Day et al., 2004), so that there would be one problem on the pre- and post-test that was directly analogous to the work that students did on the Etoys lesson (Figure 3.6). Four of the NAEP items came from the Grade 12 test and were directly related to sine and cosine functions: (1) computing sine or cosine given a right triangle, (2) identifying the x -coordinate at a given point on a sinusoidal graph, (3) identifying the amplitude and period of a given function, and (4) selecting a function that would satisfy given properties. The fifth NAEP item came from the Grade 8 exam, and it was a problem about representing a real-world context with a linear function. I included the linear problem on the pre- and post-test to gain some baseline knowledge about students' ability to translate a real-world context into a mathematical context.

- | |
|---|
| <p>1. A baby jumper is designed to let a baby sit in a secure seat that is attached to a frame and jump up and down. On one model, the seat of the baby jumper can lift the baby 60 cm off the ground, and the seat can go as low as 20 cm off the ground. It takes 1 second for the seat to cover this distance.</p> <p>a. On the axes below, sketch a graph of the baby's height off the ground as a function of time, for the first 10 seconds.</p> <p>b. Write an equation that will model the height of the baby from the floor over time.</p> |
| <p>2. Roselle is on a swing. The highest point above the ground that Roselle reaches is 9 ft, and the lowest point is 3 feet. It takes Roselle 2 seconds to travel that distance.</p> <p>a. On the axes below, sketch a graph of Roselle's height off the ground as a function of time, for the first 10 seconds. Be sure to label your axes.</p> <p>b. Write an equation that will model the height of the baby from the floor over time.</p> |

Figure 3.6. Pre- and post-test items about representing a real world context with sine and cosine.

There are both advantages and disadvantages to using released NAEP items for the pre- and post-test. The primary advantage of using released NAEP items is that the items were created by a team of experts. They have been tested and validated with large groups of students. The primary disadvantage of using released NAEP items to construct the pre- and post-test is that there were no released NAEP items that corresponded directly to the mathematical ideas that students studied during the Etoys lesson. For that reason, I had to supplement the released

NAEP items with the problems in Figure 3.6. Overall, combining the NAEP items with one original problem provided a collection of items to measure students' understanding of ideas related to sine and cosine.

The four Grade 12 test items about sine and cosine functions were multiple-choice questions. For the Grade 8 linear context problem, I used the rubric provided by NAEP and scored the item on a scale of 0—Incorrect, 1—Partially correct, or 2—Correct. For the item that I wrote, I constructed a 4-point rubric based on my examination of students' responses (see Appendix E). For each of the non-multiple choice items on the pre- and post-tests, another scorer scored 20 of the tests according to the rubric (10 randomly selected from the pre-test and 10 randomly selected on the post-test). Our reliability was over 90%.

I created two versions of the pre- and post-tests: Version A and Version B (see Appendix D). The problems on the two tests were essentially the same, with minor modifications to the numbers, differences in the order of the problems and, in multiple choice questions, the order of the choices. For example, in Version A of the test, students had a problem of determining the amplitude and period of the function $f(x) = 15\sin(7x)$. On Version B of the test students had to complete the same task, but for the function $f(x) = 10\cos(3x)$. When students took the pre-test, approximately half of the participating students in the lesson took Version A, and half of the participating students took Version B. For the post-test, the group of students who took Version A for the pre-test took Version B for the post-test. The group of students who took Version B for the pre-test took Version A for the post-test. The purpose of having two versions of the test was so that students did not take an identical test for the pre- and post-test. Having half of the participating students take each test at each stage accounted for what may have been minor, unintended differences in the level of difficulty between the two versions.

Analysis of students' pre- and post-tests. After scoring students' pre- and post-tests, I used SPSS to manage and analyze the data on students' scores. I used paired *t*-test comparisons to compare the means of students' scores. The analysis of student data through the paired *t*-tests is appropriate because the data from students' test scores followed an approximately normal distribution around the mean. In previous studies in mathematics education, researchers have used *t*-tests to compare students' scores on pre- and post-tests surrounding instructional intervention (e.g., Chizhik, 2001; Clements & Battista, 1989, 1990; Wirkala & Kuhn, 2011). My method of analysis of students' pre- and post-test work is consistent with methods that have been used previously.

I used students' scores on the pre-test to identify lower-achieving versus higher-achieving students. The maximum number of possible points on the test was 12, with 4 of those points coming from the problem about representing a periodic, real-world context. On the pre-test, students' scores ranged from 0 to 7, with a median score of 3.5. I used 3.5 as the cutoff between lower-achieving and higher-achieving students. Students earning between 0-3 points on the pre-test were categorized as low achieving, and students earning between 4-7 points on the pre-test were categorized as high achieving.

Students' Post-Lesson Interviews

After completing the Etoys lesson, 12 students participated in a one-on-one post-lesson interview. I conducted all of the post-lesson interviews, using the interview protocol in Appendix F. The interviews lasted between 15-25 minutes. The primary purpose of the interview was for students to work on a problem analogous to the Ferris wheel problem. During the interview, I asked students to solve the problem in Figure 3.7, a problem about representing the depth of water in a context of rising and falling tides. I gave students a computer with the

same Etoys set-up they had used during the lesson. I encouraged students to use Etoys to solve the problem, but I also gave students a handout so they would have the option of writing out the solution. In addition to having students work on the problem, I asked students their perspectives about the Etoys lesson. I asked what students had liked and disliked about the lesson, what had been similar to or different from work they usually did in class, and what had been difficult about the lesson. The purpose of these questions was to inform my observations of students' work based on their performance during the lesson and on the pre- and post-tests.

Throughout the day, the depth of water in Bar Harbor, Maine varies with the tides. The table shows the depths, in feet, at various times in the morning:

Time, t	Depth of water, y
Midnight	5.7
2am	10.5
4am	10.5
6am	5.7
8am	0.9
10am	0.9
Noon	5.7

Figure 3.7. The problem students worked on during the post-lesson interview.

During students' work on the problem in the post-lesson interview, I tried as the interviewer to maintain as a neutral a stance as possible, so as to gauge students' understanding of the problem from their own perspectives. While students worked on solving the problem either on paper or with Etoys, I asked students to verbalize their solution processes or ideas, so that they could communicate their thinking. In some instances, students who participated in the interviews quickly stalled in the process of solving the problem. In those cases, after giving students a couple of minutes to work without making any suggestions, I asked students more direct questions about how to solve the problem. For instance, I would ask students questions about the period or the vertical stretch of the function they would construct. My questions were

not meant to serve as clues about how to solve the tides problem, but rather to see whether any suggestion would provoke different ideas that students would talk about or include in their work on the problem. This method is consistent with principles for conducting clinical interviews in mathematics education research, which should be designed so as to elaborate as much as possible on each case (Easley, 1977). With this choice, I was able to probe students' thinking more deeply, even if they were hesitant or unable to work on the problem independently. At the same time, it is important to keep in mind that my questions may have served as cues to students to remember certain ideas or procedures that they would not have otherwise used. Although students worked on the problem independently, the nature of their thinking was shaped to some degree through our conversation.

Analysis of students' post-lesson interviews. To analyze students' mathematical work during the post-lesson student interview, I used the cK ϵ framework (Balacheff and Gaudin, 2003, 2009) to understand the conceptions of sine and cosine functions that surfaced in students' work. The cK ϵ framework for conceptions offered a way to identify, through students' actions, what conception of sine and cosine functions they used at different moments. I focused on the time period of each interview during which students worked on the problem about representing high or low tides. I segmented those portions of the interview according to when students used particular operations, representations, or control structures to complete the problem (following González, Eli, & DeJarnette, under review).

Based on my analysis of students' conceptions of sine and cosine during the in-class Etoys lesson (see Chapter 2), I had already developed a list of conceptions that had surfaced in students' work during a similar problem (see Table 3.2). I expected that similar conceptions would surface during students' work in the post-lesson interview. In my analysis of the

interviews, I performed the conceptions analysis by coding for the operations, representations, and control structures that students used to solve the tides problem. After that analysis, I labeled students' conceptions according to whether they matched conceptions I had identified in my previous study or whether they were new conceptions. The data and analysis from students' post-lesson interviews helped me to answer my second and third research question in this study. Data from the individual student interviews complemented the data from students' pre- and post-tests by revealing the details of students' thinking.

Table 3.2
Students' Conceptions of Sine and Cosine During the Etoys Lesson

Conception	Problem
Circle	Students translated the circular representation of the Ferris wheel directly to a circular curve on the Cartesian plane. This conception was not viable, because students could not graph a function that would represent the circle.
Ordered Pairs	Students plotted points in the Cartesian plane according to the convention that the x -axis would represent time and the y -axis would represent height off the ground. This is the most viable conception for plotting points.
Symbolic	Students relied on prior knowledge that a cosine function should be of the form $y = a\cos(bx) + c$. They attempted to recreate the symbolic representation, without attention to the meaning of the various components of the function.
Amplitude	Students, beginning with the graph of a function, examined how to adjust the parameters of the function to change the vertical stretch of the graph. With this conception, students reduced the complexity of the problem to give consideration to one aspect of a sinusoidal function and its graph, the amplitude.
Shift	Students examined how to adjust the parameters of the function to change the horizontal or vertical shift of the graph. Students experimented with different numbers to make a connection between the inputs and outputs of syntax.
Period	Students experimented with different coefficients to adjust the period, or the horizontal stretch, of a sinusoidal function. Students often began with a coefficient inside the function corresponding to the revolution time of the Ferris wheel, and then adjusted that coefficient according to the outputs of the syntax.
Composition	Students made connections between the contextual information about the Ferris wheel, the parameters of the sinusoidal function, and the graphical outputs. Students recognized a sine or cosine function as a composition of functions with an independent variable.

Case-Study Analysis

Data from pre- and post-tests, and from the one-on-one interviews, served as snapshots of students' understanding of sine and cosine functions before and after the lesson. Initial analysis

of students' post-tests and post-lesson interviews revealed interesting contrasts between students in terms of how they performed in those different settings. Following Flanagan (2001), I conducted a case-study analysis of five students who participated in the Etoys lesson in class and the post-lesson interview. I selected five students for the case studies based on the outcomes of their post-lesson test and their performance during the post-lesson interview. The purpose of the case study analysis was to examine differences in students' performance on the post-lesson activities and to make sense of those differences in light of the work that students had done during the Etoys lesson.

For each case study, I reviewed all of the data for each case a second time. The conceptions framework served as a guide for paying attention to students' conceptions of sine and cosine functions as they evolved over the course of the study. Because students' work in class was as part of a pair, it was not feasible to entirely disentangle individual student's understanding during the lesson from the ideas that surfaced in conversation with their group. However, when viewing the videos and reviewing the transcripts for the case study analysis, I paid specific attention to how each student participated in the lesson within their pairs. The selection of the cases is not meant to give a completely representative view of all of the potential learning activities of all the students in the class. The purpose of the five case studies is to give a more complete understanding of how individual students transferred their experiences from the Etoys lesson to work on a new problem. In addition, looking at specific cases allowed me to compare students' individual work with the work they had done in pairs.

Results

I organized my findings into three subsections, according to each of the three research questions guiding this study. First, I will give an overview of students' change in performance

from the pre-test to the post-test. Next, I will compare changes in performance between low-achieving students and high-achieving students. Finally, I will present the five case studies to examine the learning outcomes of different students through their work on the Etoys lesson.

Pre- and Post-Tests

With students' pre- and post-tests, I sought to answer my first research question: *How did students' performance on problems related to sine and cosine functions change from a pre-test to a post-test?* Table 3.3 presents means and standard deviations of pre-test and post-test scores. There were two test items on which students' scores improved statistically significantly from pre-test to post-test. Students' scores improved significantly on the item about representing a periodic real-world context with a function, $t(27)=2.36$, $p < .05$. This was the problem that was most closely aligned with students' work on the Ferris wheel problem, which suggests that students may have learned something through their work on the problem. However, the standard deviation of students' scores actually increased from the pre-test to post-test. Although students' scores improved overall, there was a larger discrepancy in students' performances on the post-test than on the pre-test.

Table 3.3
Means and Standard Deviations for Pre-test and Post-test Measures

Item Description	Pre-Test		Post-Test	
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>
Identify x -coordinate on sinusoidal graph	.36	.488	.46	.51
Identify the amplitude and period of sinusoidal function	1.79	.499	1.57	.63
Select a function with given amplitude and period	.57	.504	.50	.51
Represent a periodic real world context	.46	.793	.89	1.03*
Compute sine or cosine of right triangle	.46	.51	.75	.44*
Represent a linear real world context	.11	.42	.07	.38

Note: $n = 28$

* $p < .05$

In addition, students' scores improved significantly on a problem of computing sine or cosine, given a right triangle with two of the three sides labeled, $t(27)=2.12$, $p < .05$. This was the problem in which students were provided with a diagram of a right triangle with the legs of the triangle labeled lengths 3 and, respectively 4. Students had to compute either the sine or cosine of a labeled angle, depending on the version of the problem. The problem of computing sine or cosine of a right triangle was not directly related to students' work on the Ferris wheel problem. Based on what I could observe from video data and copies of student work, students did not use trigonometric ratios to solve the problem. However, their performance on this problem improved significantly from the pre-test to the post-test.

There were four items on the test for which students' scores did not change significantly. Students' scores increased on the item about identifying the x -coordinate of a point on a sinusoidal graph, although that change was not statistically significant. In addition, the mean scores on items about identifying the amplitude and period of a sinusoidal function, selecting a function with a given amplitude and period, and representing a linear context, all decreased slightly. These changes were not statistically significant. The purpose of including these test items, since they were not directly related to students' work on the Ferris wheel problem, was to gauge what prior knowledge students had about sine and cosine in different contexts and to see whether students' work on the Ferris wheel problem would translate to other contexts. Considering prior research suggesting students do not often establish connections among different contexts of studying sine and cosine (Thompson, 2008; Weber, 2005), it is not entirely surprising that their performance on these problems did not improve.

On both the pre-test and the post-test, students scored especially low on the item about representing a linear, real-world context using a function. The purpose of that test item was to

gain some baseline measure of students' abilities to translate a real-world situation into mathematical language, in a context that I expected to be easier than working with sine or cosine functions. Students performed better on the problem about representing a periodic phenomenon than on the problem about representing linear pattern, even on the pre-test before they had completed the Etoys lesson. Since students' scores on the linear problem were so low, and that problem was not related to sine or cosine functions, I disregarded the linear problem from further analysis.

Overall, students' performance improved on the problem most closely aligned with their work during the Etoys lesson, the problem of representing a periodic phenomenon with a sine or cosine function. In addition, students' performance improved on a problem about computing sine or cosine of an angle in a right triangle. On other test items, students' performance did not change in a statistically significant way.

Comparison of Low-Achieving and High-Achieving Students

By separating students into two groups according to their performance on the pre-test, I sought to answer my second research question: *How did students who were lower-achieving versus higher-achieving on the pre-test compare in their change in performance from the pre-test to the post-test?* Table 3.4 gives an overview of the changes in score on each of the test items. To compare low-achieving versus high-achieving students, I computed the mean change in score for each group of students for each of the test items from the pre-test to post-test.

Table 3.4
Means and Standard Deviations for Changes in Scores

Item Description	Low-Achieve		High-Achieve	
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>
Identify x -coordinate on sinusoidal graph	.19	.54	.00	.73
Identify the amplitude and period of sinusoidal function	-.31	.47	-.08	.28
Select a function with given amplitude and period	.06	.57	-.25	.45
Represent a periodic real world context	.31	.60	.58	1.31
Compute sine or cosine of right triangle	.43	.72	.08	.66

Note: $n = 28$

Overall, the differences in the change in students' scores were not statistically significant for any test item. Students in the low-achieving category seemed to improve slightly more on test items about identifying the x -coordinate of a point on a sinusoidal graph, selecting a function with a given amplitude and period, and computing the sine or cosine of an angle in a right triangle. Low-achieving students' scores improved, although slightly less than high-achieving students' scores, on the problem about representing a real world context with a sine or cosine function. Finally, low-achieving students' scores decreased more than high-achieving students' on the item about identifying the amplitude and period of a given sinusoidal function. Again, the differences in mean improvements were not statistically significant.

Cases of Individual Students' Work

After comparing students' scores on the pre- and post-test I sought to answer my third research question: *How did students' strategies on individual tasks during a post-lesson interview compare with the strategies that they used on similar tasks during work in pairs?* To compare students' work on the post-test, post-lesson interview, and their work during the lesson, I selected five different cases for analysis. I selected these five cases based on evidence from the post-lesson interview of the different ways that students used Etoys to solve the problem about high and low tides. The cases provided by Tori, Gia, Zach, Lucas, and Elizabeth illustrate

different possibilities for how students' learning about sine or cosine functions could have become instrumented through their use of Etoys. Tori and Gia were cases of students who were low-achieving students based on their work on the pre-test, while Zach and Lucas were cases of students who were high achieving based on their work on the pre-test. Elizabeth did not complete a pre-test, because she was absent from class on the day the pre-test was administered.

Tori and Gia provide examples of students who appropriated the Etoys tools in a way that allowed them to maintain relatively limited conceptions of sine and cosine functions during their work in the interview. In the cases of Tori and Gia, both students showed evidence of symbolic conceptions (Table 3.5), although they appropriated the drag-and-drop tool in Etoys to fit that conception. Zach provides an example of a student who displayed a composition conception during the post-lesson interview, although he did not use Etoys in his work. The cases of Lucas and Elizabeth illustrate students who used Etoys in ways that supported them to invoke increasingly sophisticated conceptions of sine and cosine. The cases of Lucas and Elizabeth provide contrast to the cases of Tori and Gia, who appropriated the Etoys tools to satisfy limited conceptions of sine and cosine.

Table 3.5
Case Study Students' Conceptions of Sine and Cosine Functions

Student	Conception Shifts During In-Class Lesson	Conception Shifts During Post-Lesson Interview
Tori	Circle → Ordered Pairs → Symbolic → Amplitude → Shift → Amplitude → Shift → Period → Composition	Ordered Pairs → Symbolic
Gia	Circle → Ordered Pairs → Symbolic → Amplitude → Period → Composition	Ordered Pairs → Symbolic → Shift → Symbolic
Zach	Ordered Pairs → Composition	Ordered Pairs → Composition
Lucas	Circle → Ordered Pairs → Symbolic → Amplitude → Period → Composition	Ordered Pairs → Composition → Symbolic → Composition
Elizabeth	Circle → Ordered Pairs → Symbolic → Amplitude → Period → Composition	Ordered Pairs → Symbolic → Shift → Symbolic → Amplitude → Period

I highlighted each of these cases because each of the students appropriated the tools of Etoys differently in their work during the post-lesson interview. The comparisons indicate the different ways that students' thinking may have been instrumented through their use of the technology tools. Students used the tools in Etoys for different purposes, and one student did not appropriate the tools of Etoys into his work.

Tori. Tori was an example of a student who was a low-achieving student based on the results of the pre-test. Tori earned 2 out of 12 possible points on the pre-test. Tori correctly answered the problem about identifying the amplitude and period of a given sine or cosine function. She incorrectly answered questions of identifying the x -coordinate of a given point on a sinusoidal graph, identifying a trigonometric function with given properties, and computing sine or cosine given a right triangle. Tori earned 0 out of 4 possible points on the problem of

representing a periodic, real world situation. Figure 3.8 shows Tori's work on this problem. On the pre-test, Tori had to solve the problem of representing Roselle's height off the ground as a function of time while swinging on a swing. On her graph, Tori correctly labeled what would be the appropriate minimum and maximum heights that Roselle would reach. Tori sketched what looked like a piece of a sinusoidal graph. However, Tori's graph stretched below the appropriate minimum heights that Roselle would have reached. Tori did not write an equation to represent Roselle's height off the ground. Based on her work on the pre-test, Tori did not seem to have a strong understanding of the connection between the symbolic representations of trigonometric functions and the characteristics of those functions.

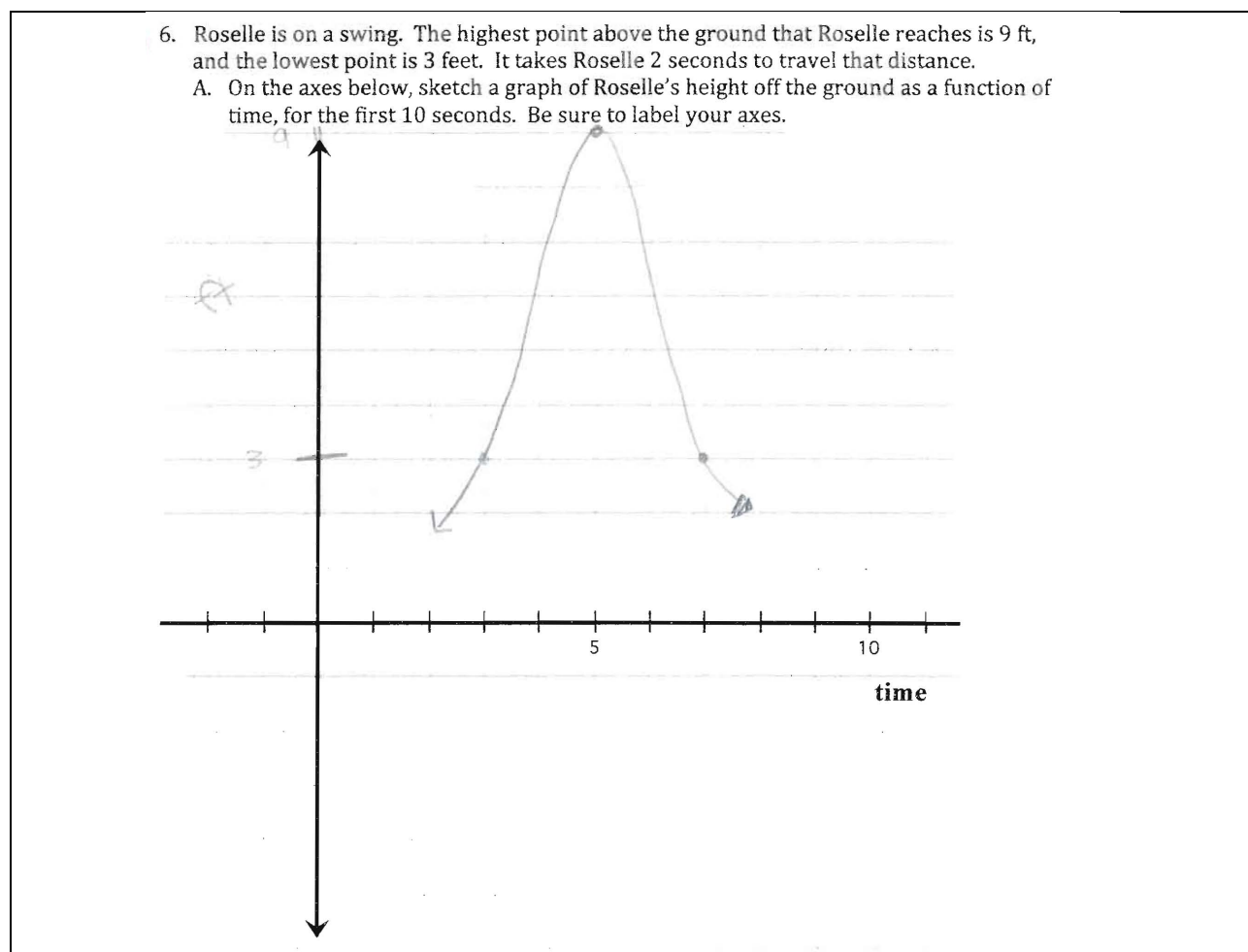


Figure 3.8. Tori's work on the pre-test item about representing with sine and cosine.

On the post-test, Tori earned 3 out of 12 points. On the problem about identifying the amplitude and period of a given function (which she had answered on the pre-test), Tori correctly identified the amplitude, but did not correctly identify the period. Tori correctly solved a problem of computing the sine or cosine of an angle in a right triangle on the post-test. Also, Tori earned 1 point on the post-test on the problem about representing a periodic, real world phenomenon using a sine or cosine function. On the post-test, Tori worked on a problem about representing the height of a baby jumper off the ground as a function of time. Tori's work on the problem can be found in Figure 3.9. Tori earned 1 point from drawing a sinusoidal graph with the appropriate maximum and minimum values. In addition to having the correct maximum and minimum values in her sinusoidal graph, Tori correctly accounted for the period of the function in her sketch of the graph in Figure 3.9. Tori earned only 1 point on the problem because she did not write an equation to represent the height off the ground. Her performance on the problem of representing a periodic, real world situation on the post-test was an improvement over her performance on the analogous problem on the pre-test.

A baby jumper is designed to let a baby sit in a secure seat that is attached to a frame and jump up and down. On one model, the seat of the baby jumper can lift the baby 60 cm off the ground, and the seat can go as low as 20 cm off the ground. It takes 1 second for the seat to cover this distance.

A. On the axes below, sketch a graph of the baby's height off the ground as a function of time, for the first 10 seconds.

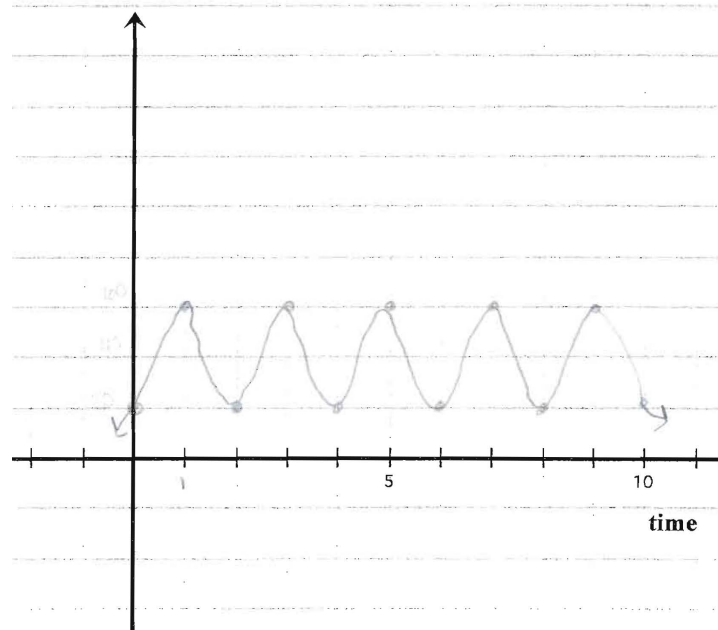


Figure 3.9. Tori's work on the post-test item about representing with sine and cosine.

During the post-lesson interview, Tori constructed a function to represent the depth of the tides at various moments during the day (Figure 3.10). The function that Tori used to represent her height off the ground, written in standard algebraic notation, was

$y = 10.5\sin(x + 12 * 3.14) + 5.7$. To summarize Tori's solution, Tori added 5.7 at the end of her function because the point corresponding to $x=0$ was located at $y=5.7$ on the graph. Tori multiplied the sine function by 10.5 to account for the maximum y -value of the points that she had plotted. Tori recognized that the period of her graph would need to be 12, and she indicated that she would need to use the number pi in her expression, for which she would use the approximation 3.14. From that information, Tori included 12 multiplied by 3.14 inside the sine expression, added to the independent variable x .

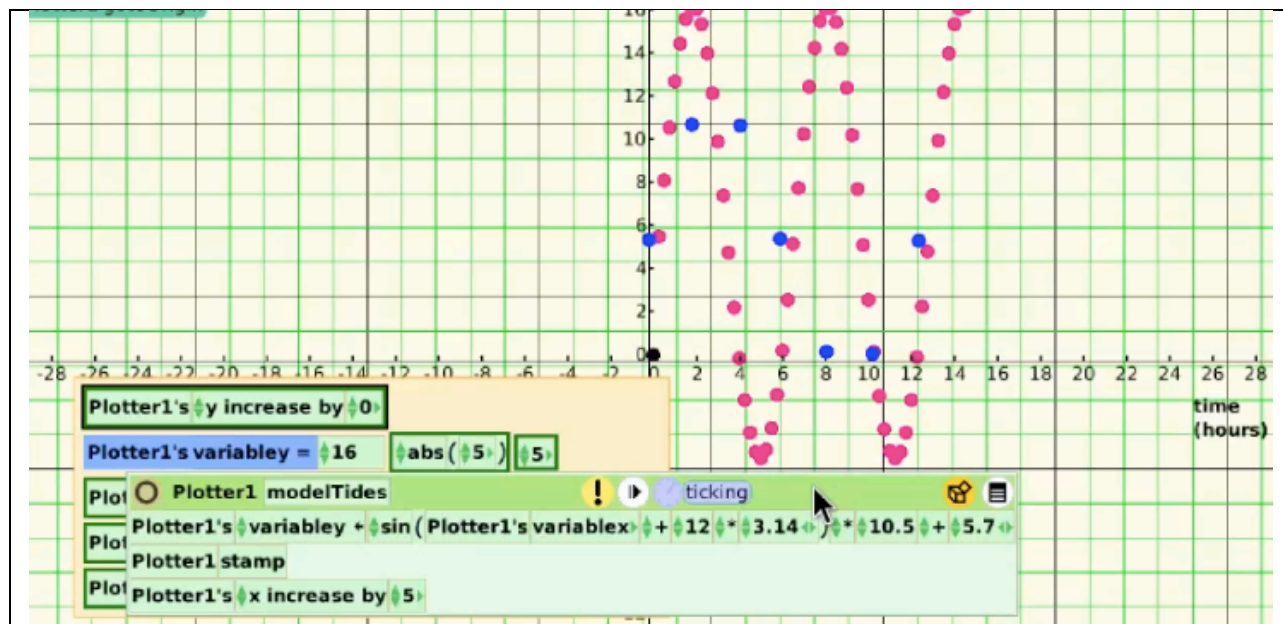


Figure 3.10. Tori's solution during the post-lesson interview.

Tori began the post-lesson interview by plotting the blue points representing the depth of the tide at various moments (see Figure 3.10). Tori immediately plotted the points according to the standard convention for a Cartesian plane. I identified her conception as an ordered pairs conception of sine and cosine. Tori's initial ordered pairs conception provided a contrast to the initial conception that Tori displayed with her partner during the Etoys lesson (Table 3.5). Although Tori and her partner began the Etoys lesson with a circle conception, during the post-lesson interview Tori began with an ordered pairs conception.

Once Tori began constructing her function with the Etoys syntax, she displayed a symbolic conception of sine and cosine that persisted throughout much of her work. Tori's first step in her work was to decide that she would use a sine function to represent the depth of the tides. After selecting a sine function, Tori wanted to enter an independent variable into her

syntax. Tori was having trouble dragging the variable tile into her syntax, so I asked Tori where she was trying to place the variable¹⁴.

AD: Where do you want to put it? Inside the sine?

Tori: Or, like, doesn't it—It usually goes after.

AD: After sine?

Tori: Yeah. Right? That's how we -

AD: You have to make a spot to put it in if you want to put it after sine.

Tori: Oh, okay.

After I suggested to Tori that she might put the independent variable inside the sine function Tori responded to my question by saying, “It usually goes after.” As is illustrated in Figure 3.11, Tori was attempting to put the independent variable after the expression $\sin(5)$. Her action was problematic in terms of her use of Etoys, because she did not indicate any mathematical operation to connect the values. Tori attempted to put the variable x after the expression $\sin(5)$ without making any relationship between the two values explicit.

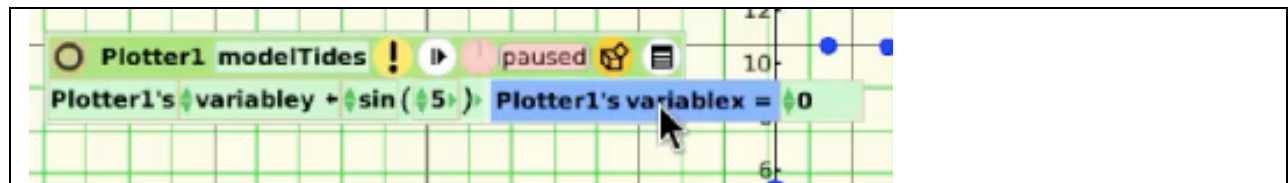


Figure 3.11. Evidence of Tori's symbolic conception of sine and cosine functions.

When I told Tori that she needed to make a place to put the independent variable if she wanted to put it after the sine function, she included an addition symbol after the sine function to create a new tile onto which she could drag the independent variable. Tori also included the number 10.5, as well as 3.14, in her script at this point, yielding the expression in Figure 3.12. At this point Tori seemed to give up with her work. Not knowing what to do next, Tori said, “I don't know. I'm confused again. This is where Sean helped me.” Sean had been Tori's partner during the Etoys lesson. To provoke Tori to continue working on the problem, I suggested that

¹⁴ In all transcripts, AD refers to the author.

she begin her work over and first plot just the function $\sin(x)$. It was at that point that Tori began working toward her final solution.

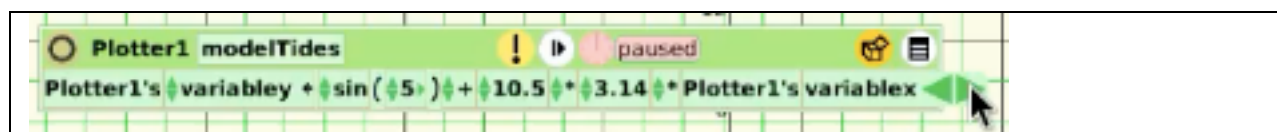


Figure 3.12. The function Tori constructed with a symbolic conception of sine.

Tori's actions indicated a symbolic conception of sine and cosine, because her operations relied on putting the symbolic representation together in a certain order, with little regard for the meaning of the representation. Tori referred to how she had usually seen sine and cosine functions written, namely as $\sin(x)$. Tori did not give attention to the meaning of the expression $\sin(x)$ as referring a function which requires an input. Instead, Tori recognized the symbolic representation as meaning that "sin" should come first and be followed by a variable. With Tori's symbolic conception, the notation " $\sin(5)$ " was equivalent to "sin", so Tori tried to enter a value after the function.

During the interview, Tori did not move beyond a symbolic conception of sine and cosine, even though she did try at different moments to use operations that she had used during the previous day's work. For example, when constructing her function during the post-lesson interview, Tori eventually decided to multiply the sine function by 10.5. I asked her why she chose to multiply by 10.5, and she responded that she remembered multiplying by 140 (the height of the Ferris wheel) during her work on the Ferris wheel problem. For that reason, Tori decided to multiply by the maximum y -value in her graph when constructing her function during the interview. During her work with her partner Sean during the Etoys lesson, Tori had in fact used a coefficient of 140 at some point in the construction of their function. Selecting that coefficient came in the context of an amplitude conception of sine and cosine functions. By

making connections between the inputs and outputs of their syntax, Tori and Sean went back and forth between a coefficient of 35 and a coefficient of 140, before finally settling on using 70 (the correct answer) for their coefficient. The difference during the post-lesson interview was that Tori did not show any evidence of an amplitude conception when she elected to multiply by 10.5. With the coefficient of 10.5, Tori noted that her graph was going too high, so I probed her to consider how she might fix that:

Tori: Now it's going too high.

AD: Mm hmm.

[Pause 3 seconds.]

Tori: Okay.

AD: So, what number do you need to change to make it go a little less high?

Tori: Um [runs mouse over different numbers in script] this one [pointing at 10.5]? I have no idea [laughing].

AD: Okay, so what does that 10.5 represent?

Tori: The highest point. Or, yeah, the highest point on the graph when the tide comes in.

When I asked Tori what number she would change to make the graph go less high, she pointed to the coefficient of the sine function, which was correct. However, Tori's selection seemed like a guess, because she pointed at all the different numbers in her script before settling on 10.5. Moreover, Tori suggested 10.5 as though it were a question instead of a statement. Following up with Tori's suggestion, I asked her about the meaning of the 10.5 that served as the coefficient of the sine function. In response, Tori told me that the 10.5 represented the highest point on her graph, according to the points that she had plotted. Tori's response suggested that she was conflating the height of the graph with its vertical stretch. Moreover, Tori's comments indicated to me that she had not made an explicit connection between the value of the coefficient and the amplitude of the graph.

Overall, based on her work on the post-test and post-lesson interview, Tori made small gains in her ability to represent real world phenomena with sine and cosine functions. From the

pre-test to post-test, Tori was better able to sketch a graph to represent periodic phenomena with the appropriate maximum and minimum values. In addition, Tori began the post-lesson interview with an ordered pairs conception of sine and cosine, which allowed her to plot the points representing the function appropriately on the Cartesian plane. During the post-lesson interview, Tori showed some indication of identifying and using the important values given in the problem. However, Tori did not move beyond a symbolic conception of sine and cosine during the post-lesson interview, as she had during her work with her partner during the in-class lesson. Instead, Tori seemed to accommodate her symbolic conception of sine and cosine to account for the requirements of the Etoys syntax. Tori still performed actions of placing numbers in certain places, but she indicated mathematical operations between them. Tori did not select the mathematical operations purposefully, but she used them to create spaces to place tiles. Tori did not seem to make connections between the inputs and outputs of her script in a way that allowed her to use a more sophisticated conception of sine and cosine.

Gia. Gia was another example of a student who was low-achieving based on her scores on the pre-test. Gia earned only 1 point out of 12 on the pre-test. On the problem about identifying the amplitude and period for a given sine or cosine function, Gia correctly identified the amplitude, but not the period. Gia did not correctly identify the x -coordinate of a given point on a sinusoidal graph, nor did she identify the correct trigonometric function given certain properties. Gia incorrectly computed the sine or cosine of an angle in a right triangle. She earned 0 out of 4 possible points on a problem of representing a periodic, real world phenomenon on the pre-test. Gia's work on this problem on the pre-test can be found in Figure 3.13. On the pre-test, Gia worked on the problem of representing Roselle's height off the ground

while swinging. Gia did not seem to recognize that Roselle's motion would be periodic or would have a maximum. Gia did not write an equation to represent Roselle's height off the ground.

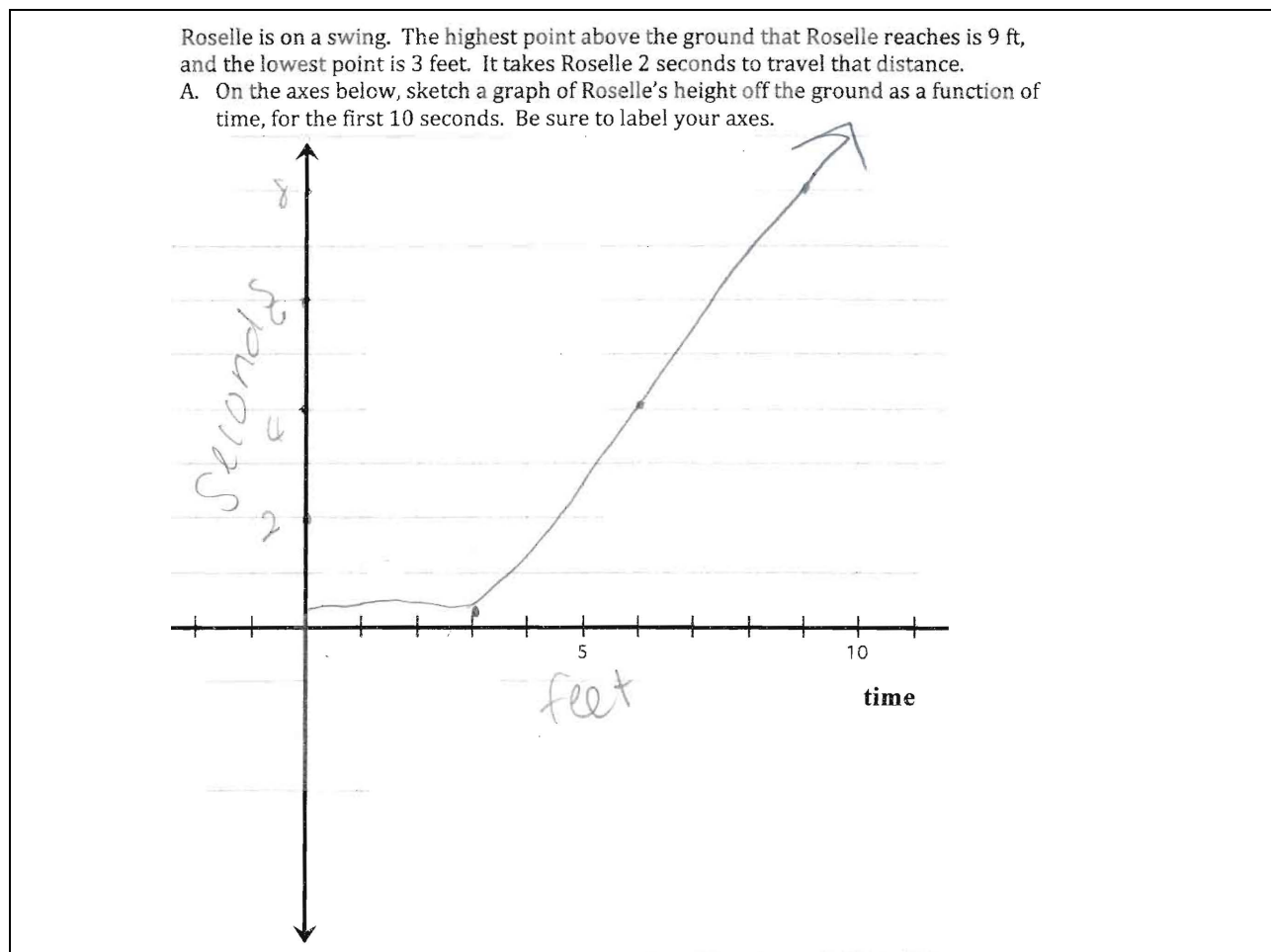


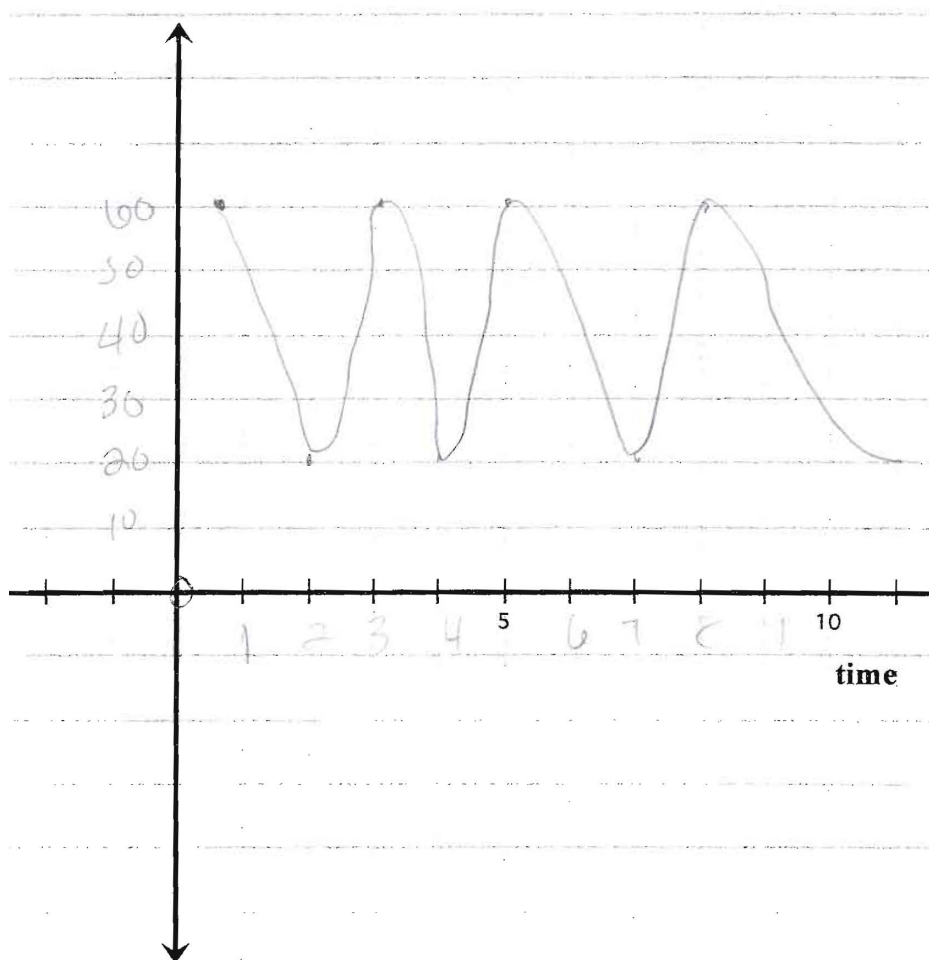
Figure 3.13. Gia's work on the pre-test item of representing with sine and cosine.

Gia earned 2 points on the post-test, compared to 1 point on the pre-test. As on the pre-test, Gia was able to correctly identify the amplitude of a given function. Gia incorrectly solved the same problems on the post-test, with the exception of the problem about representing a periodic phenomenon. On the post-test, Gia earned 1 out of 4 possible points on this problem. Gia's work on the post-test can be found in Figure 3.14. Working on the problem of representing the height of a baby jumper off the ground as a function of time, Gia sketched a sinusoidal graph with the appropriate maximum and minimum values. Gia wrote an equation to

represent the height off the ground as a function of time, although the parameters of the cosine function were not appropriate for representing the given situation. Comparing Gia's work on the problem of representing periodic phenomena from the pre-test to the post-test indicates improvement in terms of translating the real-world context to a Cartesian graph.

A baby jumper is designed to let a baby sit in a secure seat that is attached to a frame and jump up and down. On one model, the seat of the baby jumper can lift the baby 60 cm off the ground, and the seat can go as low as 20 cm off the ground. It takes 1 second for the seat to cover this distance.

- A. On the axes below, sketch a graph of the baby's height off the ground as a function of time, for the first 10 seconds.



- B. Write an equation that will model the height of the baby from the floor over time.

$$y = 60 \cos(40x + -) + 10$$

Figure 3.14. Gia's work on the post-test item of representing with sine and cosine.

During the post-lesson interview, Gia first plotted the depth of the water at various moments according to the table provided. Based on that information, Gia elected to use a cosine

function to represent the depth of the water for any given time. Gia's solution to the problem during the post-lesson interview can be found in Figure 3.15. In standard algebraic notation, the equation that Gia used to represent the depth of the water at various times was $y = \cos(y+1) + 5.7 + 1$. Gia added 5.7 to the cosine function to account for the shift up by 5.7 of the first point on the graph. Once she decided that she needed a variable for her function, Gia entered a y into the syntax. Gia did not give a clear indication of how she selected the number 1 inside the cosine function and at the end of her line of script.

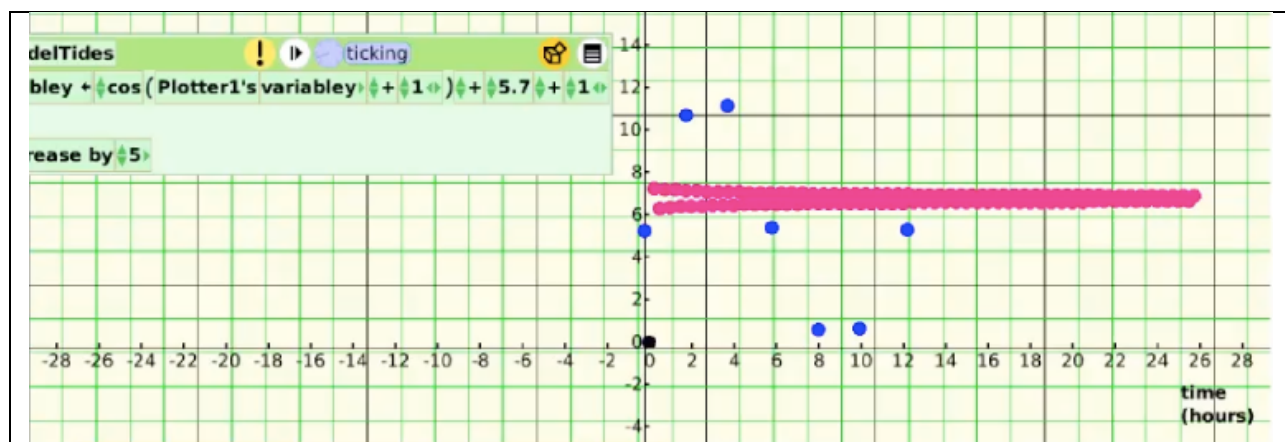


Figure 3.15. Gia's solution during the post-lesson interview.

Gia began her work with an ordered pairs conception, elicited by the problem of plotting the points to represent the depth of the water. After that, Gia's work during the post-lesson interview revealed a symbolic conception of sine and cosine functions. After she had decided she would use a cosine function, I asked Gia what else she would want to add to her script. Gia responded, "I know I need an end number." Talking about an end number indicated to me that Gia was relying on a familiar written representation of sine and cosine, in which there would generally be a constant added at the end of the symbolic representation of the function. Following this idea, Gia included several constants in her cosine function, yielding the script in Figure 3.16.



Figure 3.16. An intermediate step in Gia's work.

After Gia told me that she knew she would need an end number (indicating a symbolic conception), I asked Gia to explain what the “end number” would change about her graph. At that moment, Gia made a connection between adding a value to the end of the cosine equation and shifting the graph vertically. This connection indicated a transition from a symbolic conception of sine and cosine to a shift conception, to account for the vertical shift of the graph. Gia considered vertical shift as a feature of the graph that was connected in a specific way to the expression in her syntax. Her movement to the shift conception allowed Gia to account appropriately for the vertical shift in her Etoys syntax.

Gia's transition to the shift conception did not provoke her to move to either an amplitude or a period conception of sine or cosine. After considering the vertical shift of the function, Gia displayed a return back to the symbolic conception. This became most evident a few moments after Gia explicitly considered the vertical shift of the graph and how she would account for that in her syntax. Before arriving at her final solution, Gia defined the function in Figure 3.16. In that piece of syntax, Gia had the equation $y = \cos(5 + 1) + 5.7 + 1$. While considering what she would change next, Gia pointed at the number 5.7 with her mouse and said, “This is the amplitude, right? Cuz on this, the amplitude can't be in front of the cosine, so we put it in the back?” With this statement Gia indicated that she knew her function should contain a number next to the cosine function, and that number should account for the amplitude of the graph. Rather than placing that number in front of the cosine function, Gia knew that with the Etoys

syntax she would need to place the number after the cosine function. With that comment Gia revealed that she was still relying on a written symbolic representation of the form $y = a\cos(bx) + c$. Since her previous ways of representing sine and cosine functions did not work in the Etoys environment, Gia seemed to be trying to identify the new place to put the coefficient of the cosine function.

At this moment, Gia seemed to get stuck in her solution, so I suggested to Gia that she include a variable in her cosine function:

AD: What if you included a variable?

[Pause 2 seconds.]

Gia: Oh. [Pause 2 seconds.] It's, uh, it would be, like, $5x$ plus a number.

In response to my suggestion that Gia may need to include a variable in her function, she indicated that she could use an expression, “ $5x$ plus a number,” instead of her current expression “ $5+1$ ” (see Figure 3.16). Even after referring to $5x$, Gia put the variable y inside her cosine function. Her actions indicated that Gia was using the different variables x and y interchangeably. This was problematic because it was not consistent with the Cartesian representation of the function, which used the x variable as the independent variable. Interchanging the x and y variables suggested to me that Gia held a symbolic conception of sine and cosine. The variable served as a place holder within the symbolic representation of the function, but the variable did not carry any meaning. Once Gia realized she needed a variable inside her function, the specific variable was not important. My question to Gia about whether she should include a variable in her expression provoked her to consider using a variable, but it did not provoke Gia to change her conception. Gia still persisted with a symbolic conception following my question, but she included the variable as part of her symbolic representation.

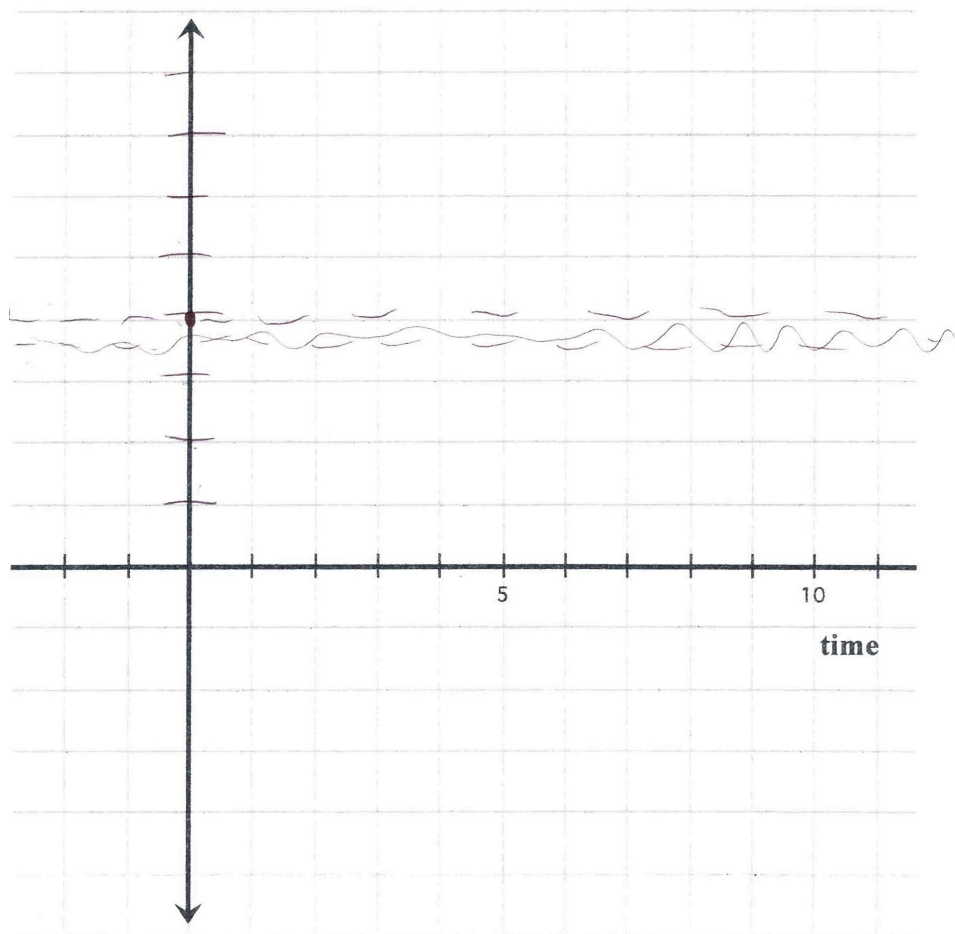
During her in-class work on the Etoys lesson, Gia and her partner had shown evidence of the composition conception of sine and cosine only during moments of scaffolding by the teacher. However, Gia and her partner briefly showed evidence of both the period and amplitude conceptions during their work in pairs with Etoys during the lesson. One might suspect that Gia had relied more on her partner during the in-class lesson, and that the conceptions surfacing during their work may have been initiated by the partner. However Gia's partner, Courtney, was also identified as a low-achieving student based on her work on the pre-test. Given that information, it seems less likely that the amplitude and period conceptions that arose during their pair work would be any more likely to be initiated by Courtney than by Gia. There are many different factors that may have contributed to Courtney and Gia's conceptions during the in-class lesson, including scaffolding from the teacher, comments from other students in the class, or Courtney and Gia's conversation and coordinated use of Etoys. During the one-on-one interview, Gia did not invoke all the same conceptions that she had during her in-class work with her partner.

Overall, Gia did not display much movement beyond a symbolic conception of sine and cosine, except for briefly when she considered a cosine function as a function with a parameter for vertical shift. The shift conception was a positive aspect of Gia's work in light of her performance on the pre-test, which indicated very limited prior knowledge of sine and cosine functions. Moreover, Gia's work on the post-test indicated a noticeable improvement from the pre-test in translating from a real world scenario into a mathematical representation. Gia's work on problems of representing periodic phenomena with sine and cosine functions improved. By the end of her post-lesson interview, Gia still maintained a symbolic conception of sine and

cosine. Moreover, she seemed to accommodate her symbolic conception to account for the syntax requirements of the Etoys environment.

Zach. Zach was a high-achieving student based on his work on the pre-test. Zach earned 5 points out of 12 on the pre-test. He correctly identified the amplitude and period of a given function, and he correctly identified a function with a given amplitude and period. Zach also correctly identified the x -coordinate of a given point on a sinusoidal curve, and he correctly computed the sine or cosine of an angle in a right triangle. On the pre-test, Zach earned 0 out of 4 possible points on a problem of representing a periodic, real world situation. Zach's work on the pre-test problem can be found in Figure 3.17. On the pre-test, Zach worked on the problem of representing the height of a baby jumper. Zach sketched a curve that looked periodic, although it did not have the appropriate maximum or minimum values. In addition, Zach indicated that he recognized that a sinusoidal function would be appropriate for representing the height at a given time. However, Zach did not use the appropriate sine or cosine function.

6. A baby jumper is designed to let a baby sit in a secure seat that is attached to a frame and jump up and down. On one model, the seat of the baby jumper can lift the baby 60 cm off the ground, and the seat can go as low as 20 cm off the ground. It takes 1 second for the seat to cover this distance.
- A. On the axes below, sketch a graph of the baby's height off the ground as a function of time, for the first 10 seconds.



- B. Write an equation that will model the height of the baby from the floor over time.

$$15\sin(x) + 40$$

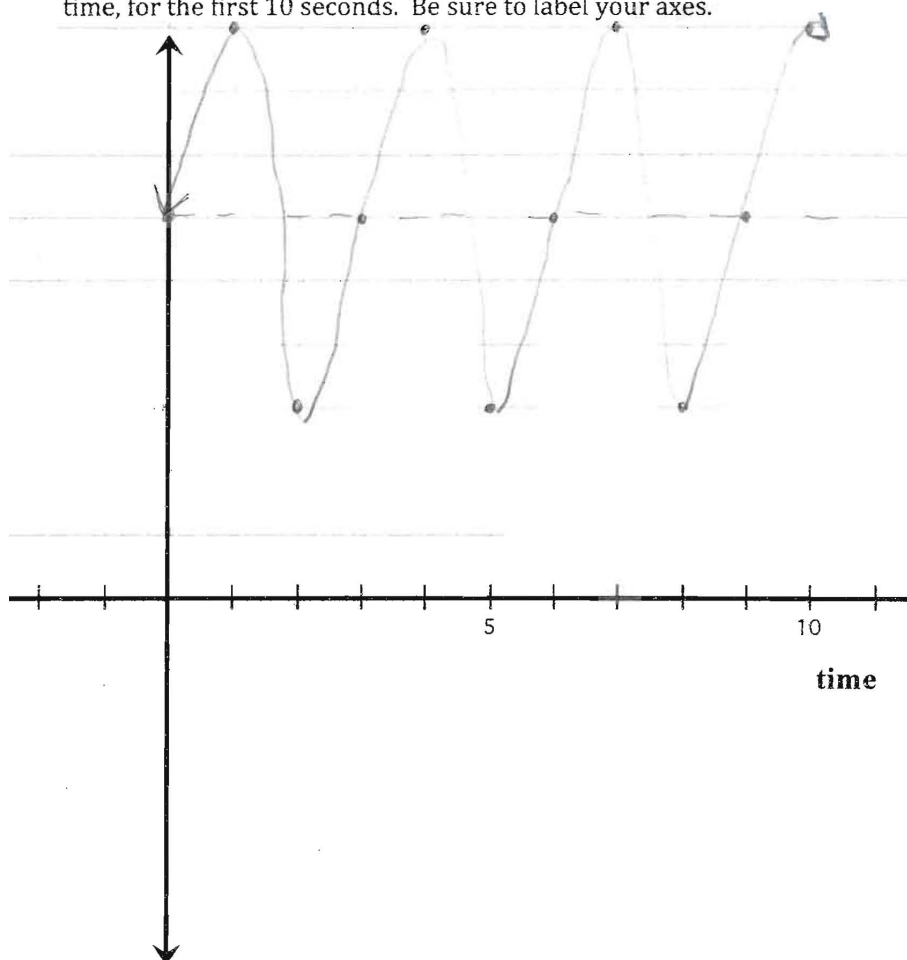
$$15\sin($$

Figure 3.17. Zach's work on the pre-test item of representing with sine and cosine.

On the post-test, Zach earned 8 points out of 12 points. He correctly answered all of the problems corresponding to the problems he had correctly answered on the pre-test. In addition,

Zach earned 3 out of 4 points on the post-test on the problem of representing the height of Roselle off the ground while swinging. Figure 3.18 includes Zach's work from the problem on the post-test. Zach drew an appropriate graph. Zach wrote an expression to represent Roselle's height off the ground with only one error. Zach used the expression $3\sin\left(\frac{\pi}{5}x\right) + 6$ to represent Roselle's height off the ground. A correct expression would have been $3\sin\left(\frac{\pi}{2}x\right) + 6$. His choice of the coefficient inside the sine function was the only error in his work.

- Roselle is on a swing. The highest point above the ground that Roselle reaches is 9 ft, and the lowest point is 3 feet. It takes Roselle 2 seconds to travel that distance.
- A. On the axes below, sketch a graph of Roselle's height off the ground as a function of time, for the first 10 seconds. Be sure to label your axes.



Write an equation that will model the height of Roselle off the ground over time.

$$3 \sin\left(\frac{\pi}{5}x\right) + 6$$

Figure 3.18. Zach's work on the post-test item about representing with sine and cosine.

Zach provided an interesting case during the Etoys lesson because he chose to work primarily without the use of Etoys, both during the Etoys lesson in class and during the post-lesson interview. During the post-lesson interview, Zach used Etoys to plot points representing

the depth of the water in the tides problem, as in Figure 3.19. Working on that problem, Zach used an ordered pairs conception of sine and cosine. Once Zach had plotted the points using Etoys, he switched to using paper and pencil to work on the problem.

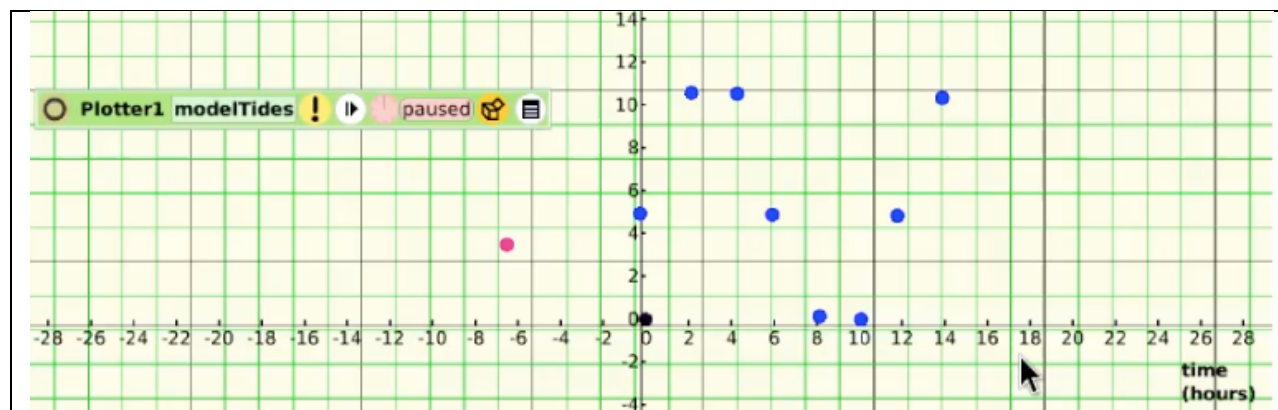


Figure 3.19. Zach's work in Etoys during the post-lesson interview.

Working with paper and pencil, Zach used a sine function to represent the depth of the water (Figure 3.20). Zach used the equation $y = 5.7 \sin\left(\frac{\pi}{6}x\right) + 5.7$. To summarize Zach's solution to the problem, he added 5.7 to the end of the function to account for the vertical shift up from 0 to 5.7 of the initial point. Zach used a coefficient of 5.7 in front of his sine function, because he approximated the distance from the midline of the graph to the top of the graph to be around 5.7. Zach used a coefficient of $\frac{\pi}{6}$ inside his function to account for the period of the function, which was equal to 12.

High Tide and Low Tide: Throughout the day, the depth of water in Bar Harbor, Maine varies with the tides. The table shows the depths, in feet, at various times in the morning:

Time, t	Depth of water, y
Midnight	5.7
2am	10.5
4am	10.5
6am	5.7
8am	0.9
10am	0.9
Noon	5.7

$$y = 5.7 \sin\left(\frac{\pi}{6}x\right) + 5.7$$

2:00

Figure 3.20. Zach's work with paper and pencil on the post-lesson interview.

After Zach had finished plotting the points, he displayed a composition conception as he began to construct the function to represent the depth of the tide. Since Zach was working with paper and pencil, I asked him about each step as he wrote down the various components of the function. A discussion about the amplitude and vertical shift of the function displays Zach's composition conception. I asked Zach about his choice to use 5.7 as the coefficient of the sine function:

AD: So, how do you get this number, the 5.7, in front of the sine?

Zach: Well, because it has to go below and above the midpoint.

AD: Okay.

Zach: The midpoint would be going through all of these [gesturing to the blue points corresponding to $x=0, 6$, and 12 in Figure 3.17].

AD: So where's the midpoint?

Zach: Uh, 5.7.

In our conversation, Zach indicated that he chose the coefficient of his sine function to make the graph stretch above and below its midline, although Zach seemed to be using the term "midpoint" to refer the midline of the graph of the function. He chose 5.7 as the coefficient of his sine function, "because it has to go below and above the midpoint." Zach then indicated that the midline would be represented by the points on the screen with y -coordinate 5.7. With his

comments, Zach indicated that he used the distance from the extreme values to the midline of the graph to compute the coefficient of his function. That operation was critical for revealing Zach's composition conception, because Zach gave explicit attention to the relationship between the amplitude and vertical shift when constructing the symbolic representation of the function. Zach seemed to miscalculate the distance from the midline to the maximum and minimum values to be equal to 5.7, but the computational error did not take away from his conceptual understanding of the solution.

Zach's work on the post-test and in the post-lesson interview reflected his in-class work during the Etoys lesson. During the Etoys lesson, Zach and Jalisa spent some time experimenting with the Etoys software, but not really working towards a solution to the problem. When Ms. Alexander told the pair that they could work on the problem with paper and pencil, Zach abandoned his use of Etoys and solved the problem on paper. During the in-class lesson, Jalisa seemed to follow Zach's lead while he did most of the work on paper and on the computer. When it came to working on the post-lesson interview, Zach did not attempt to construct his function using Etoys. He immediately solved the problem with paper and pencil, and he displayed a composition conception of sine and cosine in his work. Zach may have benefited from a dynamic representation of his function, namely in that it may have provoked him to correct the error in the amplitude of his graph. However, the error in his amplitude seemed to be a computational error, and not an indication of a change in conception of sine and cosine.

Lucas. Lucas was an example of a student who was a high-achieving student based on his scores from the pre-test. Lucas earned 4 points out of 12 on the pre-test. He correctly answered questions about identifying the amplitude and period of a sinusoidal function, identifying a trigonometric function with given properties, and identifying the x -coordinate of a

given point on a sinusoidal graph. On the pre-test, Lucas earned no points on the problem of computing sine or cosine of an angle in a right triangle. In addition, Lucas earned 0 out of 4 possible points on the pre-test item of representing a periodic phenomenon with a sinusoidal function. Lucas's work on the pre-test item of representing a real world periodic phenomenon can be found in Figure 3.21. Lucas sketched a graph with the appropriate minimum and maximum values. However, his graph was not sinusoidal, but was instead a collection of line segments. The period of his graph was appropriate for the context. Lucas did not write an equation to represent the height off the ground as a function of time. Based on his work on the pre-test, Lucas seemed to have a firm grasp on the ideas related to sine and cosine that he had recently studied in Algebra 2. However, Lucas did not apply these ideas to solve a real world problem.

6. A baby jumper is designed to let a baby sit in a secure seat that is attached to a frame and jump up and down. On one model, the seat of the baby jumper can lift the baby 60 cm off the ground, and the seat can go as low as 20 cm off the ground. It takes 1 second for the seat to cover this distance.
- A. On the axes below, sketch a graph of the baby's height off the ground as a function of time, for the first 10 seconds.

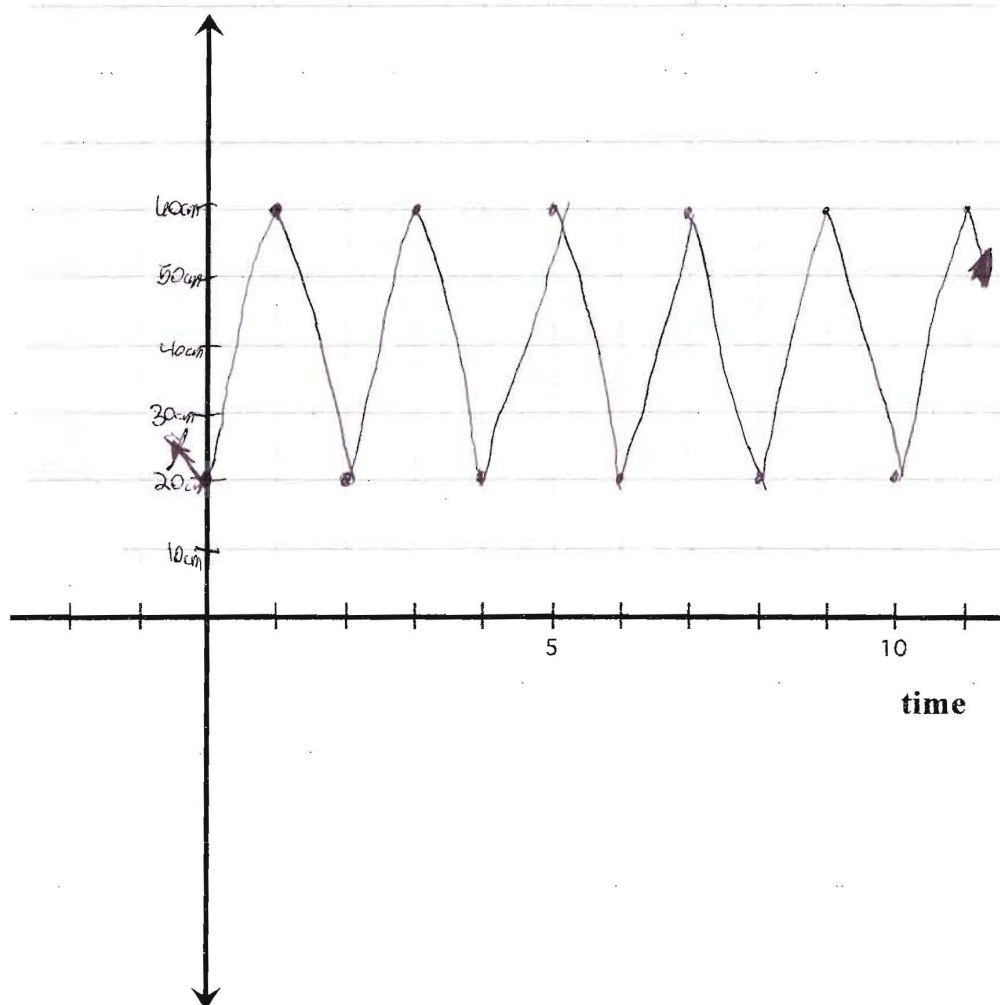


Figure 3.21. Lucas's work on pre-test item of representing with sine and cosine.

Lucas earned 5 points out of 12 on the post-test. On the post-test, Lucas correctly solved the problem of computing sine or cosine of a right triangle, in addition to correctly answering the same problems he had answered correctly on the pre-test. Lucas's work on the post-test problem about representing a real world situation with a sine or cosine functions gives an interesting comparison with the pre-test (see Figure 3.22). Lucas still earned 0 out of 4 possible points on

the post-test item, however his solution was quite different than his solution on the pre-test. On the post-test, Lucas drew a sinusoidal curve. The values of the graph ranged from 3 to 8. In order to earn at least one point for having the correct graph, Lucas would have needed to have the correct maximum and minimum values, which would have been 3 and, respectively, 9. In addition, the mid-line of Lucas's graph was not in the correct place.

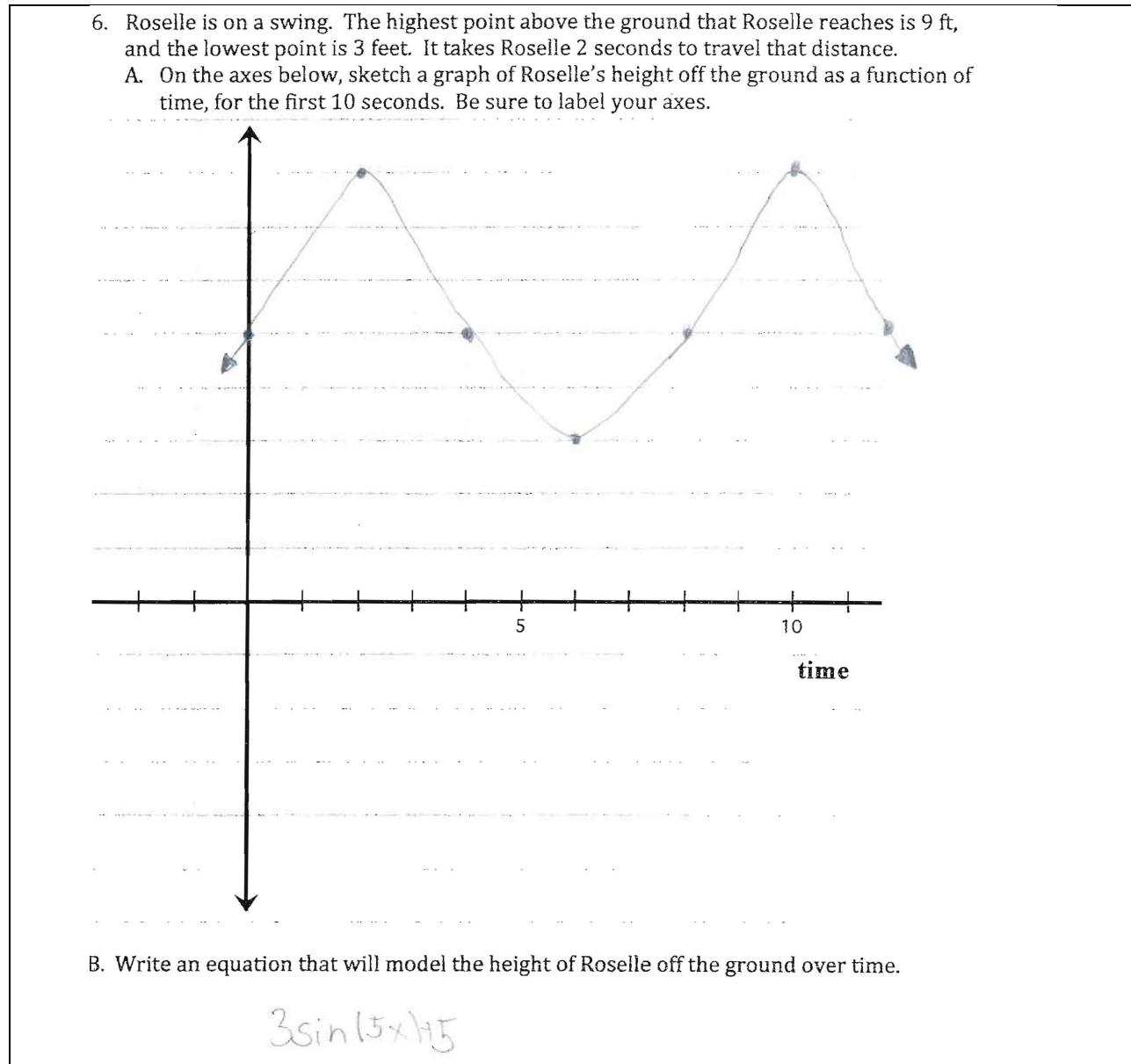


Figure 3.22. Lucas's work on the post-test item about representing with sine and cosine.

On the post-test, Lucas included an expression that would represent the height of Roselle off the ground as a function of time. Lucas used the expression $3\sin(5x) + 5$ to represent Roselle's height as a function of time. Lucas was correct in using a positive sine function to represent the graph. A coefficient of 3 in front of the sine function was appropriate for representing the situation, although the coefficient did not match the graph that Lucas had sketched. The coefficient of 5 inside Lucas's expression was incorrect, as was the vertical shift of 5. However, $y=5$ was where the midline of Lucas's graph was located, so the vertical shift in his expression was consistent with the graphical representation he constructed.

That Lucas earned 0 out of 4 points on the post-test item does not entirely reveal Lucas's ability to represent the situation using a sinusoidal function. Looking more closely at Lucas's work, it seems possible that Lucas miscounted the vertical units on his graph, which made the maximum value of the graph at $y=8$ instead of $y=9$. The coefficient of 3 in front of the sine function was actually the appropriate coefficient, and if Lucas only counted up from the midline to the maximum (and ignored the distance from his midline to the minimum) he would have come up with that coefficient. As I mentioned previously, Lucas identified the midline of his graph (indicated by the points he plotted) at the line $y=5$, which was consistent with him adding 5 to the end of his expression. The coefficient of 5 inside the sine function was the only component of Lucas's expression that did not have a clear connection to his graph. Overall, it seems that one error in constructing his graph (not reaching the appropriate maximum height) may have caused the inconsistencies in Lucas's solution that resulted in him earning 0 out of 4 possible points on the problem.

Lucas's work on the post-lesson interview more thoroughly explains his ability to solve a problem about representing a real world situation with a sine or cosine function. Figure 3.23

displays Lucas's final solution to the problem of representing the depth of water during high and low tides. Written in standard algebraic notation, Lucas used the function

$$y = 5.7 \sin\left(\frac{3.14}{6}x\right) + 5.7$$

to represent the depth of the water at time x , where 3.14 was an

approximation for π . The correct equation to represent the situation using a sine function would

$$y = 5.6 \sin\left(\frac{3.14}{6}x\right) + 5.7$$

, so Lucas's solution was very close to correct. Had Lucas's

work during the post-lesson interview been scored according to the same rubric as his work on

the post-test, Lucas would have earned 3 out of 4 possible points on the problem.

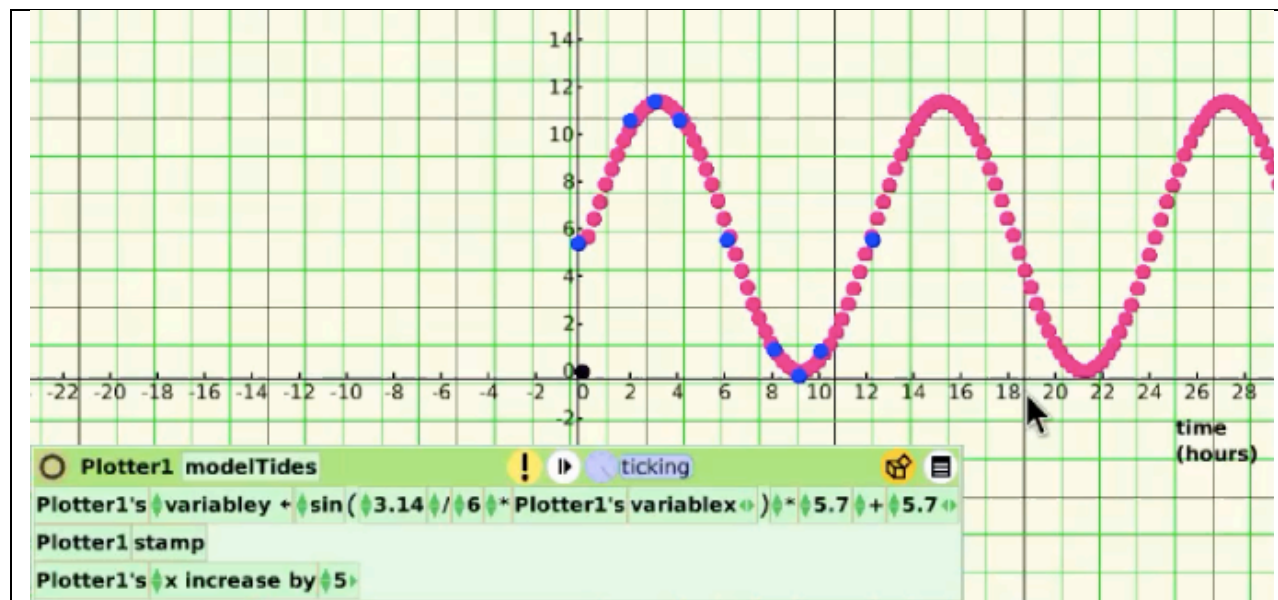


Figure 3.23. Lucas's solution during the post-lesson interview.

Lucas did not immediately establish the solution to the tides problem found in Figure 3.23. During the interview, Lucas transitioned from an ordered pairs conception, to a composition conception, to a symbolic conception, and then back to a composition conception of sine and cosine functions. The ordered pairs conception of sine and cosine functions was elicited by the problem of plotting the blue points to represent the depth of the water at various moments. After plotting those points, Lucas first invoked a composition conception when determining the

coefficient of the cosine function. Lucas initially computed the coefficient of the cosine function to be 4.8 (see Figure 3.24). The value 4.8 came from subtracting 5.7 (the y -coordinate of the midline of the graph) from 10.5 (the y -coordinate of the maximum value Lucas had plotted). With this operation, Lucas indicated the connection between the vertical shift and the amplitude of the graph of a sinusoidal function, and how those two things contribute together to determine the coefficient of the function. The coefficient of 4.8 was incorrect, because 10.5 was not actually the maximum value of the graph, it was simply the maximum value that Lucas had plotted. Lucas corrected his computational error later in his work. Most important at this point in his work was Lucas's composition conception of sine and cosine, which accounted for the relationship between the amplitude and the vertical shift of the sine function.

Lucas transitioned to a symbolic conception, which became apparent in Lucas's work when he tried to account for the period of the graph in his script. Lucas noted, "The period is 2π over b , and b is 6. So, 2π over 6 is π over 3." With that calculation, Lucas first entered $3.14/3$ inside his function (see Figure 3.24).

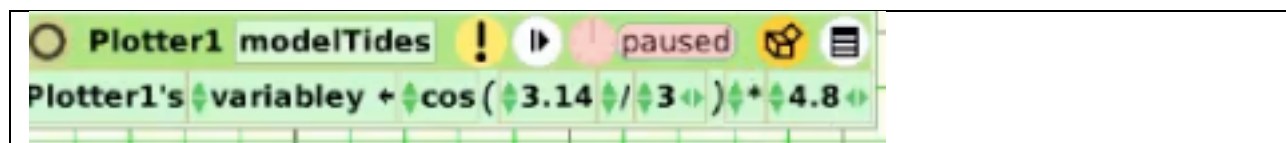


Figure 3.24. Evidence of Lucas's symbolic conception of sine and cosine functions.

The symbolic conception was made apparent by two aspects of Lucas's work. First, Lucas used a formula, "period is 2π over b " without indicating in any way what the meaning of b was. Given that Lucas and his classmates had prior knowledge of writing sinusoidal functions written as $y = a \sin(bx) + c$, I assumed that b referred to the coefficient of the independent variable inside the function. However, without making any explicit connection to the meaning of b , but invoking an algebraic equation that he had used in the past, Lucas's conception was

closely tied to his prior knowledge of the written symbolic representation of a sine function. The other evidence of Lucas's symbolic conception was the missing independent variable inside the cosine function in Figure 3.24. When defining the function, Lucas accounted for the period of the function but did not make a connection between the period and the independent variable. The operation of only entering a number inside the function indicated the symbolic conception, which only required having certain things in certain places. With this conception, the variable x did not carry any significant meaning, but was more of a place holder inside the trigonometric function. When Lucas ran his script and plotted a straight line, I probed him to consider what had happened:

Lucas: Why is it making a line?

AD: Why is it making a line? Cosine's supposed to make an up-and-down curve.
[Pause 2 seconds while Lucas studies his script.]

Lucas: Where did I mess up? [Pause 1 second.] Oh, x . Where's the x ?

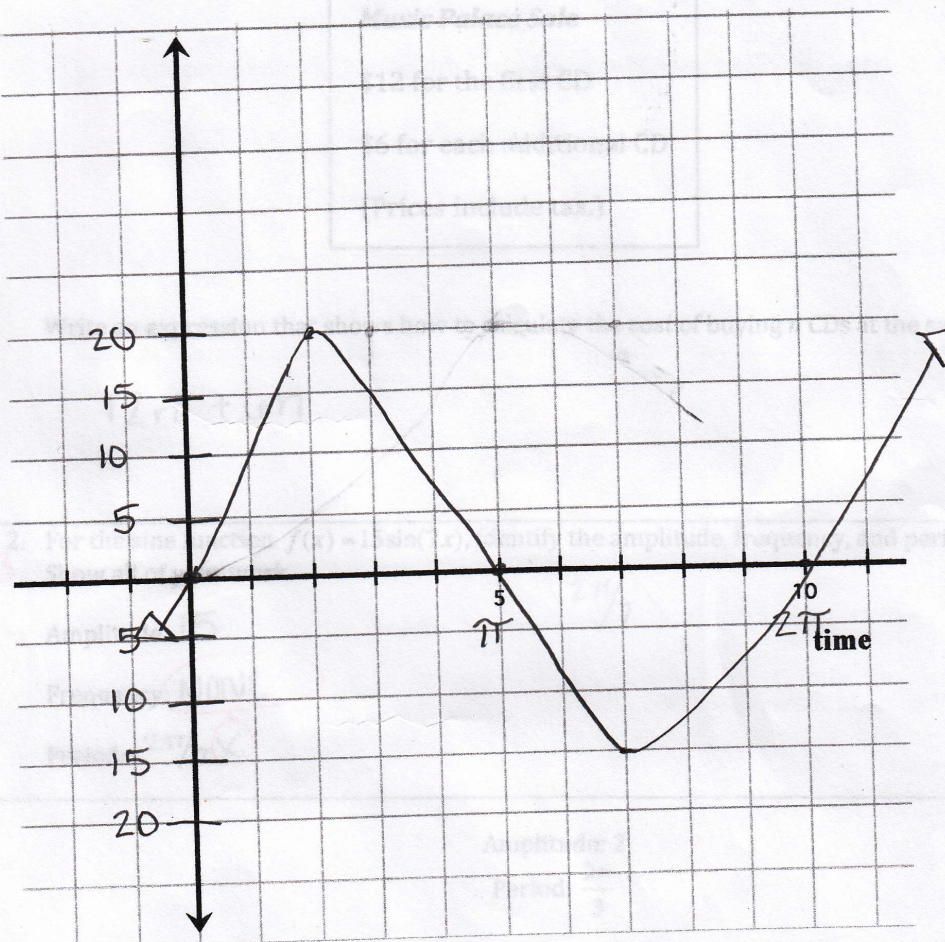
Lucas had been able to satisfy the measure of control of the symbolic conception, namely that the cosine function looked like it should, even though it was missing an independent variable. However, when Lucas ran his script, and observed the connection between the input and output, he recognized that his solution was not correct. It took only a few seconds of consideration for Lucas to move beyond the symbolic conception and to account for the independent variable in his function. During his work in the post-lesson interview, Lucas did not pass through the intermediate amplitude, period, or shift conceptions. Once Lucas observed the output of his script from Figure 3.24, Lucas moved directly to a composition conception of sine and cosine functions. Lucas made connections between the independent variable, height, period, and vertical shift of the graph. He plotted almost exactly the appropriate graph to represent the depth of the water.

Two things were important about Lucas's work on the post-test and the post-lesson interview. First, comparing the post-test to the post-lesson interview shows that the post-test did not explain entirely Lucas's understanding of representing real-world situations using sine and cosine functions. Based on the standardized rubric, Lucas earned 0 out of 4 possible points on the problem aligned with his work on the Ferris wheel problem. Although Lucas's work on the post-test revealed several inconsistencies, all of those inconsistencies may have stemmed from a single error. Lucas's work on the post-lesson interview illustrated much better his understanding of sine and cosine functions. Using the tools of Etoys, Lucas made connections between the inputs and outputs of his script, and he examined the correctness of the function he constructed and adjusted it accordingly. Although his first script yielded a graph that did not represent the depth of the water appropriately, Lucas corrected his script to come up with a function that contained only a minor error.

A second important observation in Lucas's work on the post-lesson interview was that Lucas transitioned directly from a symbolic conception to a composition conception of sine and cosine functions. Lucas did not pass through, or go back and forth between, amplitude, period, or shift conceptions during his work in the interview. This observation is important in light of the fact that, during the Etoys lesson, Lucas and his group members made 4 transitions between symbolic, amplitude, period, and shift conceptions before showing evidence of a composition conception. Lucas's work during the post-lesson interview suggests that, although he initially relied on a conception closely tied to his prior knowledge of sine and cosine, he was able to move to a more viable conception relatively quickly. This gives evidence of Lucas's learning through the Etoys lesson, in spite of the fact that his score on the post-test did not improve.

Elizabeth. Elizabeth was the only student who participated in the post-lesson interview but did not participate in the pre-test for the Etoys lesson. I include Elizabeth as a case here because her work provides an example of a student who used Etoys during the post-lesson interview in a way that vastly improved her initial solution to the tides problem. Elizabeth earned only 1 point out of 12 on the post-test. On the post-test, Elizabeth correctly identified the amplitude of a given sine function. She incorrectly answered all the other multiple choice problems. Elizabeth earned 0 out of 4 possible points on the post-test problem of representing a real world situation with a sinusoidal function (Figure 3.25). On the post-test, Elizabeth worked on the problem about representing the height off the ground of a jumper. Elizabeth sketched what looked to be a sinusoidal curve, however her curve did not have the appropriate maximum or minimum values. The function that she wrote to represent the height of the jumper included a sine function, but it did not include any of the appropriate parameters.

6. A baby jumper is designed to let a baby sit in a secure seat that is attached to a frame and jump up and down. On one model, the seat of the baby jumper can lift the baby 60 cm off the ground, and the seat can go as low as 20 cm off the ground. It takes 1 second for the seat to cover this distance.
- A. On the axes below, sketch a graph of the baby's height off the ground as a function of time, for the first 10 seconds.



- B. Write an equation that will model the height of the baby from the floor over time.

$$20 \sin(2\pi x)$$

$$\frac{2\pi}{b} = \frac{1}{1} \quad \frac{2\pi}{1} = \frac{1}{b}$$

$$2\pi = b$$

Figure 3.25. Elizabeth's work on the post-test item about representing with sine and cosine.

Elizabeth's work during the post-lesson interview is interesting because she initially revealed a very limited understanding of sine and cosine functions, similarly to what she reflected with her solution to the post-test. However by using Etoys to work through the

problem, and moving among different conceptions of sine and cosine, Elizabeth made substantial improvements to her solution to the tides problem. Elizabeth's final solution to the problem can be found in Figure 3.26. Elizabeth used the equation $y = \sin\left(\frac{6.28}{11.4}x\right) * 6 + 5.7$ to represent the depth of the tides. To summarize Elizabeth's solution, she added 5.7 at the end of her function to account for the vertical shift of the graph. Elizabeth determined the coefficient of the sine function, as well as the coefficient of the independent variable inside the sine function, through a method of trial and error.

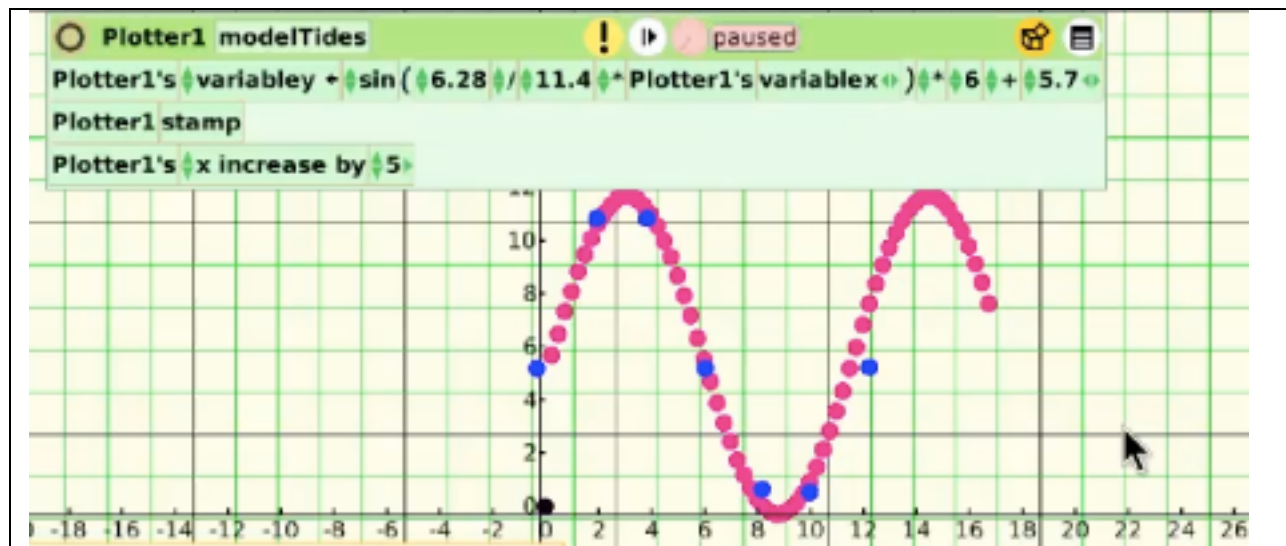


Figure 3.26. Elizabeth's solution to the tides problem during the post-lesson interview.

Elizabeth's first step towards solving the tides problem was to plot points representing the depth of the tide at various moments. During that work Elizabeth invoked an ordered pairs conception of sine and cosine. After that, Elizabeth went through a process of several steps to determine the appropriate graph that would run through the points she had plotted. Elizabeth decided that she would use a sine function to represent the situation. After that decision, Elizabeth seemed stuck in her attempt to solve the problem. After several seconds of silence, I asked Elizabeth to think about the period of the function:

AD: What do you think is the period of this?
 Elizabeth: Um [pause 4 seconds] I'm not sure.
 AD: What do you think of when I say the period of a sine graph?
 Elizabeth: I just think of the equation, and the 2π over b .

When I asked Elizabeth what she thought was the meaning of the period of a sine graph, she provided me with the formula, 2π divided by b . Again based on students' previous work in Algebra 2, I assumed that b referred to what would be the coefficient of the independent variable inside the sine function. With her statement, Elizabeth revealed a symbolic conception of sine and cosine. Even though I probed Elizabeth to consider the period of the function, this probing did not push Elizabeth beyond a symbolic conception. Elizabeth relied on an algebraic formula for understanding the period of the function, but she did not have a way to connect that formula with the graphical representation of the problem, or with the real world context. This symbolic conception led Elizabeth to use numbers from the problem to try to apply the formula, coming up with a coefficient of $\frac{5}{10.5}$ in her function (see Figure 3.27).

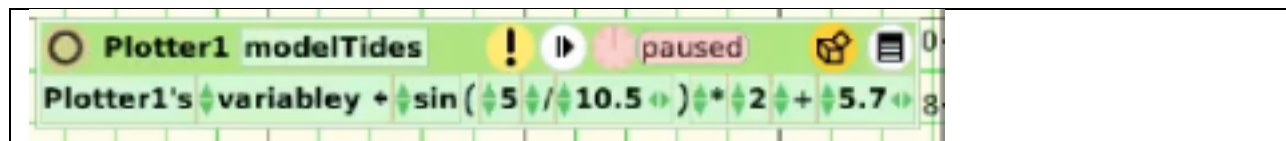


Figure 3.27. An intermediate step in Elizabeth's work.

Elizabeth moved back and forth among the different parameters of her sine function, eliciting different conceptions of sine as she did. After thinking for some time about the period of the function, Elizabeth accounted for the vertical shift (Figure 3.27). Elizabeth added 5.7 to the end of her sine function and commented, "So then that's the vertical shift." By adding 5.7 to the function, Elizabeth indicated a shift conception, a conception considering a sinusoidal function as a function with a parameter for vertical shift. Elizabeth did not consider the vertical

shift in relation to the other parameters of the problem, but she did make a connection between the symbolic representation and the placement of the blue points she had already plotted.

After invoking the shift conception, Elizabeth returned to a symbolic conception, satisfied that her function looked like a typical sine function, with numbers inside, a coefficient, and a constant added to the end. When Elizabeth ran the script in Figure 3.27, her script plotted a straight line. Observing this, Elizabeth noted that there was a problem in her graph:

Elizabeth: [Whispering] Something's wrong.

[Pause 5 seconds.]

AD: So look at maybe your rule for -

Elizabeth: Oh, we need an x .

After several seconds of silence, I began to suggest to Elizabeth that she consider the rule she had used to assign a value to the y -coordinate of the plotter. Elizabeth interrupted my statement to point out that she needed to include an independent variable in her function. It is not clear exactly what provoked Elizabeth to remember that she would need to use a variable to construct her function. It is possible that my suggestion to look at the rule she constructed was enough of a cue to remind Elizabeth. It is also possible that she remembered from the previous days' work that she would need to include a variable, or that the straight line prompted Elizabeth to consider how to make a graph that would increase and decrease. Based on this realization, Elizabeth adjusted her script to include the variable x , and she ran the script again (Figure 3.28). At this point, Elizabeth moved on to consider how she would adjust her script to make the graph overlap all of the blue points she had plotted.

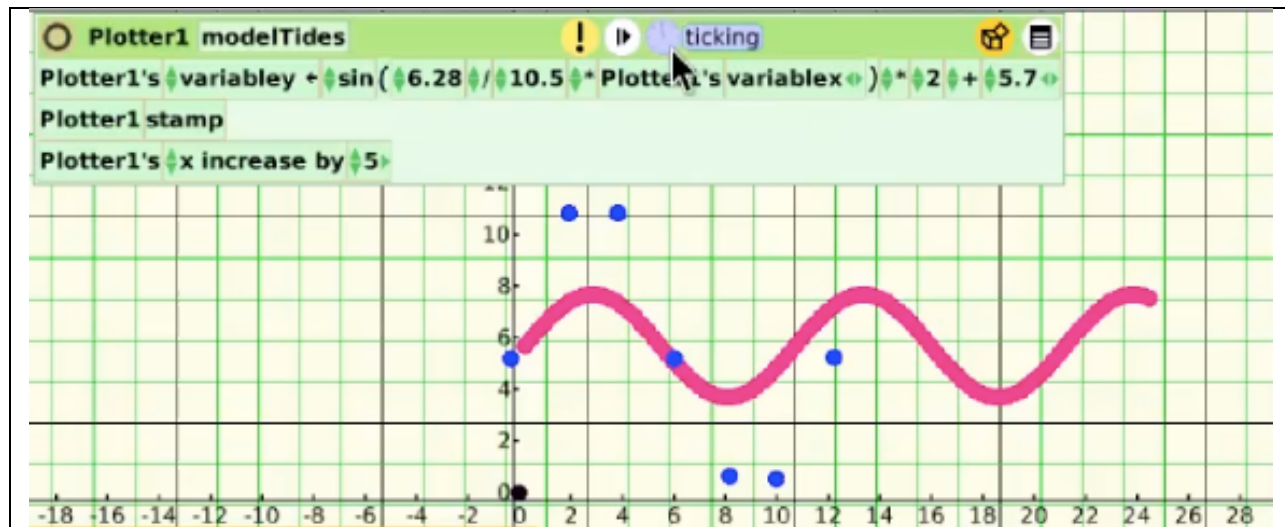


Figure 3.28. Elizabeth's work when she included an independent variable in her script.

To provoke Elizabeth to continue working towards a solution to the problem, I asked her to reflect on her solution:

- AD: So what's not quite right about the graph?
 Elizabeth: The amplitude.
 AD: Mm hmm. So what did you say is the amplitude?
 Elizabeth: Um, I think I said 5.7. Yeah. [Pause 1 second.] No, 2, I mean.

After she identified what part of the function accounted for the amplitude of the graph, Elizabeth used the scanning tool in Etoys to adjust the amplitude until it the graph was stretched vertically enough to reach the maximum and minimum points that she had plotted (Figure 3.29). Elizabeth did not make any comments indicating an explicit connection between the midline and the extreme values to compute the coefficient of the sine function, and therefore I did not identify Elizabeth as invoking a composition conception of sine. However, Elizabeth took an important step when she experimented with the parameter that would affect the amplitude of the graph. Elizabeth displayed an amplitude conception when she made connections between the input and output of her syntax, thus connecting the symbolic and graphical representations of the function.

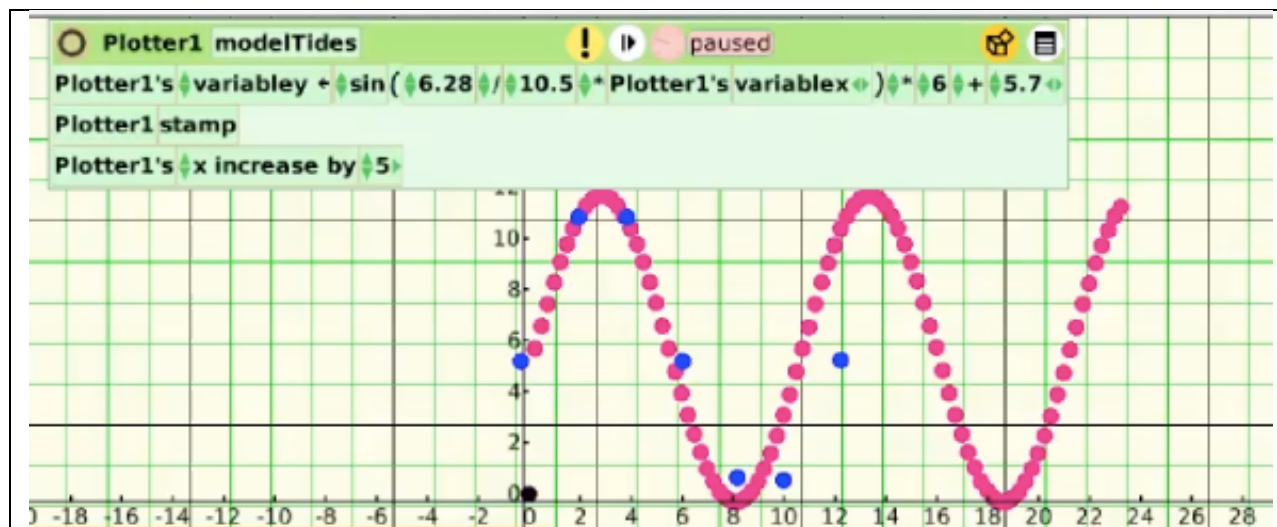


Figure 3.29. Elizabeth's solution to the tides problem during the post-lesson interview.

Last, Elizabeth moved to a period conception of sine and cosine as she finished her solution to the problem. Similarly to how she used the scanning tool to adjust the vertical stretch of the graph, Elizabeth used the scanning tool in Etoys to adjust the horizontal stretch of her graph. Again, Elizabeth made connections between the symbolic representation of her function, provided by the syntax, and its graphical output. Elizabeth did not make connections between the coefficient inside her function and the formula she had suggested earlier in the interview. Nor did Elizabeth make connections between the coefficient and the real world context of the problem. For these reasons, I would not suggest that Elizabeth showed any indication of a composition conception of function. Elizabeth did, however, make the critical connection between the two representations concerning the period of her graph.

Elizabeth provides an important case of students' work with Etoys because her solution to the problem during the post-lesson interview improved substantially. Given her solution on the post-test, it seems that Elizabeth's work during the interview could be credited at least in part to the way she used the tools in Etoys to move among conceptions of sine and cosine. When I asked Elizabeth to consider aspects of her solution such as the period, amplitude, and the

function she was using, Elizabeth responded by adjusting her Etoys script with the drag-and-drop and scanning tools. Elizabeth appropriated these tools in a deliberate way that allowed her to make connections between the symbolic and graphical representations of the shift, amplitude, and period of the function. Although Elizabeth did not reach a point of connecting all the various parameters to each other, Elizabeth took important steps towards using more sophisticated conceptions of sine and cosine to solve the problem than the one she started with.

Discussion

Measuring Student Learning Through Pre- and Post-Tests

The comparison of students' scores from pre- to post-test revealed few statistically significant gains. This is not entirely surprising, given that the pre- and post-tests measured students' learning after a relatively short intervention, where students worked on only one problem. Even given the short duration of the intervention, students' scores improved statistically significantly on the test item that was aligned with the work they did during the Etoys lesson. Although that improvement was small, it was important in consideration of the potential for students to learn something through work on one problem with the use of technology tools over two days. Students who were classified as low-achieving according to their scores on the pre-test did not have gains in performance that were significantly different than students who were classified as high-achieving.

There are two primary limitations to the findings based on students' work on the pre- and post-tests. First, there were several results that were not statistically significant, both in changes in score from pre- to post-test as well as differences in improvement between low-achieving and high-achieving students. This may suggest that the test items used to measure students' prior knowledge and their knowledge gained from the Etoys lesson did not allow for an analysis that

was fine-grained enough to make comparisons between students or before and after the lesson. This limitation is primarily a factor of the limited availability of standardized test items related to the topic of sine and cosine functions, especially how these functions can be used to represent real world contexts. Since trigonometric functions present a unique challenge for students in the study of function, there is opportunity for further research to develop resources that will allow for better examination into students' learning of this topic.

In addition, I identified students as low-achieving or high-achieving depending on their scores on the pre-test. Students' work on the pre-test focused specifically on their prior knowledge of sine and cosine functions and ratios. For that reason, using students' pre-test scores to sort them according to achievement allowed me to compare students based on their knowledge of content that was directly aligned to what they would learn during the Etoys lesson. Given that no student earned more than 7 points on the pre-test (out of a possible 12 points), dividing students as low-achieving or high-achieving was entirely relative to the other students in the classes. No student performed exceptionally well on the pre-test, meaning no student had mastered all of the relevant prior knowledge, including trigonometric ratios, sinusoidal graphs, and the parameters of trigonometric functions.

While results of the pre-test limited the comparisons I could make between students from this study, it does suggest an opportunity for future work. Literature suggests that students have many, and possibly conflicting, experiences with sine and cosine as ratios and functions (e.g., Thompson, 2008; Weber, 2005). Rather than combining all this prior knowledge to make up one composite score of prior achievement, it may be beneficial to examine whether there is some target knowledge that will support students to be more or less successful learning about sine and cosine functions through a programming environment. For example, if students who first study the unit

circle are better prepared to transition to trigonometric functions (Weber, 2005), then perhaps comparing students according to their mastery of the unit circle representation would shed more light on who is best positioned to benefit from the use of a programming environment to represent real world contexts. One could hypothesize that students who particularly struggle with graphical representations of sinusoidal functions could benefit from the connections between dynamic representations, especially in light of prior research suggesting Computer Algebra Systems may support the development of mathematical concepts (Heid, 1988). In my future work, I see potential to identify what prior knowledge will best support students to learn about the concept of sine and cosine functions through their use of a programming environment.

Even given the limitations of the study, it is encouraging to see that students had the potential to learn about sine and cosine by working for two days on an open-ended problem with the use of Etoys. Research suggests that when students have opportunities to work on open-ended problems, they tend to be well-equipped to transfer their learning to novel situations (Boaler, 1998; Boaler & Staples, 2008). From a practical standpoint, there may be a concern that teaching and learning mathematics through problem solving requires too much time devoted to a smaller numbers of problems, which may not provide students with enough practice to sustain new knowledge. In the case of the Etoys lesson, students' work on one problem over the course of two days gave them some foundational knowledge that they were able to take with them and apply to a new, similar problem on the post-test.

Hiebert et al. (1996) discussed knowledge in mathematics as the *residue* (Davis, 1992) that remains after students complete an activity or solve a problem. Residue is the understanding that results as a by-product of a problem solving activity. Through their work on the Ferris wheel problem, students developed insights into the structure of the problem and strategies for

solving the type of problem represented by the Ferris wheel problem. After their work on one problem, students were able to apply this understanding in a new context. After working on one or two other similar problems, one could suspect that students' understanding of representing real world situations with sine or cosine functions would compound and result in a more robust conceptual understanding of this topic.

Examination of Cases of Students

Looking at the cases of Tori, Gia, Zach, Lucas, and Elizabeth complemented the analysis of students' pre- and post-tests to illustrate how students learned about sine and cosine through their work on the Ferris wheel problem. Considering the model of Instrumented Activity Situations (Verillón & Rabardel, 1995) for the case of each student during the post-lesson interview, I could see that students' activities were instrumented in different ways through their use of Etoys (Figure 3.30). The case of Zach is represented across the bottom of the model, as a student who engaged with the concept of sine and cosine functions without appropriating the tools of the Etoys environment. The other four students in the post-lesson interview are represented around the top of the model. They illustrate that, for students who initially invoked symbolic conceptions, their use of Etoys resulted in different outcomes. Students can maintain symbolic conceptions, even after they appropriate the tools of Etoys in their work. Or, students can appropriate the tools of Etoys in ways that improve their thinking towards more viable conceptions for solving a problem. Examining how five different students compared in their work from pre-test to post-test, and on the post-lesson interview, revealed differences in how students appropriated the tools of Etoys in ways that supported learning.

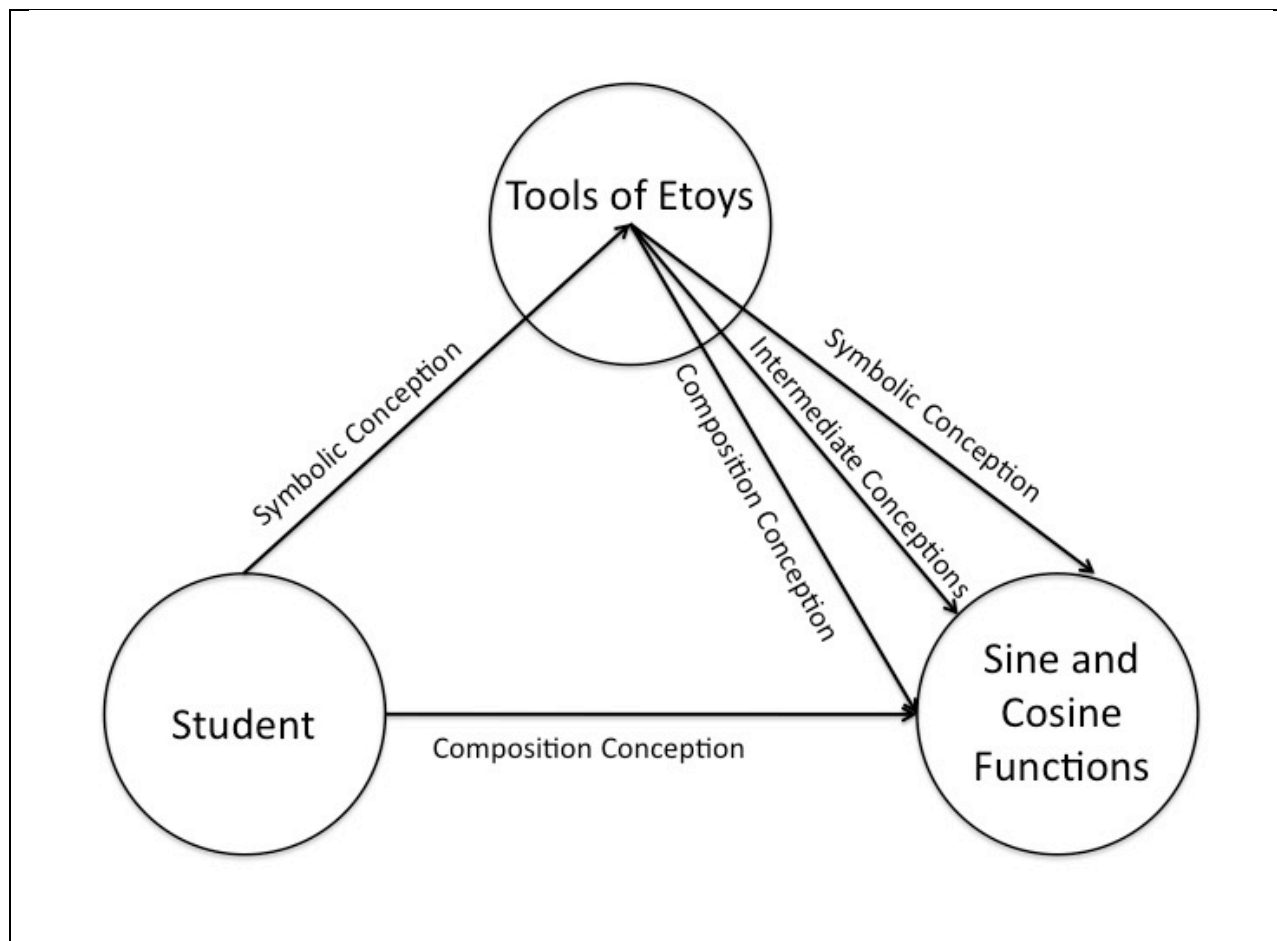


Figure 3.30. Different ways student thinking was instrumented through their use of Etoys during the post-lesson interview.

Returning to the notion of instrumented activity, Verillón and Rabardel (1995) suggested that the ways individuals appropriate technology tools change their thinking. There were two different ways in which students' instrumented activity was especially apparent during the post-lesson interviews, one of which had a positive impact on students' conceptions and one that did not support students to develop more sophisticated conceptions. In the cases of Lucas and Elizabeth, their use of the tools of Etoys allowed them to invoke more sophisticated conceptions of sine and cosine that allowed for more correct solutions to the problem. While on the post-test, Lucas earned 0 out of 4 possible points, Lucas correctly solved the problem during the post-lesson interview and indicated a composition conception of sine and cosine functions.

Completing the problem with Etoys, Lucas made connections between the inputs and outputs of his script that allowed him to construct an appropriate function after his symbolic conception yielded the wrong solution. Lucas used the scanning tool in Etoys to make connections between the inputs and the outputs of his syntax. Making connections between the different representations of the function within the Etoys environment supported Lucas towards overcoming an overly simplistic conception of sine and cosine. Lucas began his work constructing a function with a symbolic conception of sine and cosine. Through his activity with Etoys, Lucas moved on to a composition conception.

Elizabeth made similar gains through her instrumented activity with Etoys. Although Elizabeth did not display a composition conception during the post-lesson interview, Elizabeth used the scanning tool in Etoys in a way that helped her move beyond a symbolic conception. Elizabeth initially engaged with the tides problem with a symbolic conception of sine. By using the tools of Etoys, Elizabeth moved to intermediate conceptions, including the shift, period, and amplitude conceptions. With those conceptions, Elizabeth made critical connections between the symbolic and graphical representations of her function through the use of Etoys. Given that Elizabeth earned 0 points on the post-test problem of representing a real world situation, it seems that Elizabeth's activity with Etoys was critical for her to make substantial progress towards solving the problem.

Earlier research on computer-programming environments has examined the potential of programming environments to exploit the connections between symbolic and visual representations of mathematics (Clements & Battista, 1989; Edwards, 1997; Hoyles & Healy, 1997). The work of Lucas and Elizabeth indicated that connecting the inputs of syntax (a symbolic representation) with the outputs (a graphical representation) supported both students to

learn about sine and cosine functions. During the post-lesson interview, Lucas and Elizabeth illustrated how their appropriation of the Etoys tools supported this learning. By integrating what they had learned with Etoys from the Ferris wheel problem and applying it in a new context, Lucas and Elizabeth were able to develop solutions to the tides problem.

Tori and Gia's instrumented activities were apparent in the ways that they appropriated the drag-and-drop tools of Etoys to work within their symbolic conception of sine and cosine. Tori gave evidence of this activity by indicating mathematical operations between values in her syntax without regarding the meaning of those operations for the outputs of her syntax. In the post-lesson interview, when Tori wanted to put a variable x "after" the sine function, she included an addition symbol in between to make a place to put the variable. Tori included mathematical operations in her syntax that did not provide a correct solution to the problem (e.g., the addition inside her function), but that allowed her to create spaces to put tiles. By learning how the drag-and-drop tools of Etoys worked, Tori revised her symbolic conception of sine and cosine to accommodate the restrictions of the Etoys syntax.

Gia gave a slightly different example of appropriating the tools of Etoys to fit with a symbolic conception. According to Gia's prior knowledge, the coefficient of a sine or cosine function always went in front of the function. Working with Etoys, Gia learned that these coefficients would need to go after the function in the Etoys syntax. Gia revised her representation of a sine function from $a \sin(x)$ to $\sin(x) \times a$, so that she could accommodate the Etoys syntax restrictions. The symbolic representation looked slightly different, but Gia still relied on a symbolic representation in her work. Although they used slightly different operations, the result of Tori's and Gia's instrumented activity was similar. Both students initially engaged with the tides problem with a symbolic conception, and both students integrated

Etoys in their work while still maintaining a symbolic conception. This provided a stark contrast to the work of Lucas and Elizabeth, who improved their conceptions through their use of Etoys, and Zach, who simply chose not to use Etoys.

Although I did not identify any new conceptions in students' work during the post-lesson interviews, the conceptions apparent in Tori and Gia gave examples of a new way that students invoked the symbolic conception. Previously, during their in-class work, students' symbolic conceptions had relied on a written symbolic representation of sine and cosine that they had used in their previous work. The syntax requirements of the Etoys software pushed students to move beyond a symbolic conception. Perhaps because by the time of the interview students had grown more proficient with the Etoys syntax, they identified how a sine or cosine function should look in Etoys. Instead of the written symbolic representation that students had used in their previous work, Tori and Gia relied on an "Etoys symbolic" way of thinking about sine and cosine functions.

A particular challenge identified in the teaching and learning of algebra is developing the ability to use symbols appropriately (Chazan, 2000; Drijvers, 2000; Yerushalmy, 1999; Yerushalmy, 2006; Yerushalmy & Chazan, 2002). During students' work in class, using Etoys pushed students to give explicit attention to the meaning of the symbols they used in their work (see Chapter 2). However, based on some students' work during the Etoys interview, it seems that using a computer-programming environment was insufficient to help all students use symbols meaningfully and to make connections between representations. This result suggests a potential area for future research, to examine how students' use of a programming syntax can promote more meaningful use of symbolic representations.

Conclusion

Overall, students' work on the Etoys lesson showed positive outcomes in terms of their ability to transfer their understanding of the Ferris wheel problem to work on a similar problem. Although students' scores improved, the distribution of students' scores on the post-test suggests that the overall improvement may be attributed to greatly improved work among a subset of students, on a subset of problems. This means that some, although not all, students learned something through their work on the Etoys lesson that they were able to transfer to a new setting. Based on the cases of five students, it is likely that some students benefitted from their use of Etoys more than others. Although some students in the class appropriated the tools of Etoys in ways that support improved mathematical understanding, other students appropriated the same tools in ways that did not ultimately change their understanding of mathematics.

Based on this study, it still remains to be seen whether previously lower-achieving students serve to benefit more or less than higher-achieving students from the use of a programming environment for learning algebra. Previous research in this area has shown both benefits and pitfalls in struggling students' use of computer-programming environments (e.g., Healy & Hoyles, 2001; Yerushalmy, 2006). There are opportunities to develop measures of student understanding that will assess how students' prior knowledge of sine and cosine are relevant for their learning about trigonometric functions through their use of technology tools. Ideally, students should use the tools of a programming environment in ways that allow them to understand sine and cosine functions both *with* and *against* their own conceptions (Herbst, 2005). Evidence of how students' uses of Etoys challenged their initial conceptions show promise for how students can improve their understanding of mathematical concepts through the use of technology tools.

Methodologically, this work illustrates how statistical measures of student learning can be complemented by more in depth examinations of students' conceptions. Pre- and post-tests of students' knowledge of sine and cosine indicated that students learned something through their work on the Ferris wheel problem, but they did not explain why some students benefitted from their use of Etoys more than others. By applying the cKc framework (Balacheff & Gaudin, 2002, 2003) to an analysis of one-on-one interviews, I was able to identify more clearly the differences in the ways that students used the tools of Etoys as part of the operations, representations, and control structures of their conceptions. Combining the two methods provided a more complete picture of students' learning than what could have been achieved through either method in isolation.

This study contributes to research examining how students can learn mathematics through their use of technology. There is a strong tradition of scholarship suggesting that students' use of technology supports novel and improved ways of thinking about mathematics (e.g., Heid & Blume, 2008). Technology environments, such as computer-programming environments and dynamic geometry environments are expansive, and there is much to be gained from understanding how students appropriate subsets of tools in these environments for learning mathematics. To fully understand the value of technology tools, it is important to unpack not only how students use those tools, but also how students' use of the tools become part of the ways they think about mathematics. With this work, I have illustrated that the different ways students appropriated the tools of the Etoys environment likely contributed to differences in students' learning outcomes on a more general scale.

This study also has implications for the practice of teaching and learning mathematics with technology. Based on theoretical and empirical work, mathematics education policy

documents call for finding meaningful ways to integrate technology into the teaching and learning of mathematics (NGAC, 2010; NCTM, 2000). For teachers, this implies a responsibility to implement tasks in which students will learn mathematics through their use of technology tools. This study suggests that students can learn fundamental mathematical concepts even as they learn how to use the tools of a technology environment. However, teachers must be cognizant of how students come to use those tools in their mathematical work. If instruction can support students to use technology tools in ways that challenge initial conceptions, students can learn mathematics with understanding that will translate to new problems and new contexts.

CHAPTER 4:
MATHEMATICS STUDENTS' POSITIONING PATTERNS
DURING PAIR WORK WITH ETOYS

Group and pair work have become regular features of the work that many students do across mathematics classrooms and schools in the United States. Though there is no recent quantification of the prevalence of group work in mathematics education, many new curricula call for collaboration as a core part of students' work in class, including, for example, the *College Preparatory Mathematics* series (e.g., Dieteker et al., 2006, 2007), the *Connected Mathematics Project* (e.g., Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998), and the *Interactive Mathematics Program* (e.g., Fendel, Resek, Alpher, & Fraser, 2008). The leading professional organization for mathematics educators in the United States emphasizes the importance of students working together at least some of the time in mathematics classrooms (NCTM, 2000). Most recently, the *Common Core State Standards for Mathematics* proposed that students should develop abilities to communicate mathematical arguments to their peers, persevere in solving problems, and make sense of the problem solving strategies of others in school mathematics (NGAC, 2010). Although students have opportunities for these activities in multiple settings, including whole-class and individual work, working with peers provides a particularly useful setting for students to engage in important mathematical practices.

In technology rich settings, students who work together with peers need to manage their interactions with peers as well as the shared use of the technology resources. To study students' collaboration in settings where students work on computers, it is essential to take into account the students, the mathematical task, and the software that students use for the mathematical task (Hoyles, Healy, & Pozzi, 1994). The ways that students communicate arguments to their peers is

likely to look unique in a setting where students use the language of a computer environment for talking about mathematics (Healy, Hoyles, & Sutherland, 1990; Hoyles, Healy, & Sutherland, 1991; Hoyles & Sutherland, 1989). Students using dynamic technology tools tend to formulate more mathematical arguments than students who do not have access to technology tools (Smith, 2011). Research in computer science has identified that, when individuals work in pairs on a programming activity, they are better able to produce solutions to unfamiliar tasks than when working alone (Canfora, Cimitile, Garcia, Piattini, & Visaggio, 2007; Williams & Kessler, 2003). Drawing on research from computer science and mathematics education, it is clear that individuals who collaborate on programming activities benefit from that collaboration. However, the nature of students' collaboration when working with computers should be examined in order to know how to support this activity in mathematics classrooms.

In this study I examine students' work in pairs, and opportunities for productive problem solving, in the context of an Algebra 2 course. Students worked in pairs over the course of 2 days to solve a problem about representing a real world phenomenon with a sinusoidal function. To work on the problem, students used a computer-programming environment called Etoys. With my research questions, I sought to understand students' interactions through the lens of how students positioned themselves towards their peers. In addition, I tried to make a connection between students' positioning practices and their collaborative problem solving efforts. Much of the research and theory that informs this study has been conducted in the context of more typical group work (with groups of 3 or more) without the presence of technology. I have drawn on this existing body of research to frame my own study, though I have made distinctions between group and pair work, and technology versus non-technology environments, where necessary.

Research Questions

This study is guided by RQ3: *How did pairs of students' patterns of positioning support or inhibit the collective problem solving of the group?* My overarching research questions can be further sub-divided into three questions that frame this study:

1. What positions did students enact through their work with their peers during pair work at the computer?
2. How did students enact and change positions through their talk?
3. What types of positioning practices supported or hindered students in their problem solving processes during pair work?

My first two research questions ask the “what” and “how” questions about students’ positioning practices. Specifically, my first question seeks to identify patterns in the positions that students enacted towards their peers. My second question targets how students enacted those patterns, and also how students changed positions. My third research question examines more closely the interaction between students’ positioning practices and the nature of their mathematical problem solving. Taken together, these questions were designed to help me understand the interpersonal aspects of the ways students worked in pairs, as well as how students’ interactions were related to their problem solving activities. Prior research on the ways in which group work may support or inhibit students’ learning provides useful background knowledge for this study, and positioning theory offers a framework for understanding how students make moves to position themselves towards their peers during pair work.

Review of Research on Group and Pair Work in Mathematics Classrooms

In typical group work settings, students work together in groups ranging from three to five people. Textbooks in particular call for this sort of group work, as many textbooks aligned

with current standards for mathematics education encourage students to be divided into four different roles within their groups, including roles such as *facilitator*, *task manager*, *resource manager*, and *recorder* (e.g., Dietiker, Baldinger, Cabana, Gulick, Shreve, & Lomac, 2006; Fendel, Resek, Alpher, & Fraser, 2008). Assigning students to different roles during group work is largely based on the principles of Complex Instruction, which uses roles to promote participation of all members of a group and give groups guidelines for how to accomplish the work of a particular task (Cohen, 1994a). Much of the research on students' collaboration has examined group work with three or more people (for a comprehensive review, see Esmonde, 2009a). In settings where students work together with computers, students often work in pairs (e.g., Hoyles, Healy, & Pozzi, 1994). Since students engage in many similar behaviors during group work and pair work, for example asking questions and giving explanations to peers, I draw on both areas of research to inform this study.

The purposes and positive outcomes of group work in mathematics classes have been well-documented in mathematics education literature (see Cohen, 1994b; Webb & Palincsar, 1996 for reviews). From the perspective of students' mathematical achievement, there are a variety of findings regarding students' behaviors during group work and how those behaviors support mathematical learning. The activities of asking for help and giving help to peers are strong predictors of future achievement. Students tend to learn more when they give elaborated help to their peers, compared to when they provide direct answers to questions with no elaboration (Webb, 1989, 1991). When students elaborate on their explanations, they are able to clarify material, fill in gaps in understanding, and make connections between different representations. For helping behavior to be most useful, help-givers need to give thoughtful, elaborated explanations, and help-seekers need to make use of such explanations and apply them

immediately to the task at hand (Webb, Farivar, & Mastergeorge, 2002). Receiving help from peers supports students' mathematical understanding and also provides motivational support for students.

Other studies have identified individual behaviors that lead to positive outcomes in students during group work. For example Azmitia (1988) found that lower-achieving students benefited from guidance from experts and observations of experts' strategies. Students who work together to complete tasks benefit from joint decision-making regarding the strategies they would use to complete a task (Chizhik, 2001). Students benefit from jointly monitoring their progress towards a solution (Schoenfeld, 1989) and having opportunities to explain their thinking (Cohen, 1994b). In general, group work seems to give students opportunities to engage in behaviors that support their own learning, which they would not be able to engage in if they were working alone.

In addition to supporting achievement, classroom environments that regularly employ group work reflect improved relationships between students from diverse backgrounds, and more diverse groups of students identifying with the practice of mathematics (Boaler, 1998; Boaler & Staples, 2008; Gutiérrez, 2002). Group work supports students to engage with mathematics while participating in the social processes of a classroom (Cobb, Boufi, McClain, & Whitenack, 1997). During group work, students can support one another to engage in mathematical practices, for example the practice of generalizing mathematical ideas (Ellis, 2011). In addition, group work settings can create opportunities for students to overcome traditional power structures in classrooms, by asserting themselves within small groups of their peers (Esmonde & Langer-Osuna, 2013). This research suggests that group work in mathematics classrooms has benefits that extend beyond traditional achievement tests. Traditional achievement tests can

indicate what mathematical understanding students have gained of certain topics, but group work also benefits students through the processes by which they gain that understanding. Through interactions with peers, students gain interpersonal skills that support their mathematical activities.

Even given the positive outcomes of group work, students' success within this type of activity cannot be taken for granted. There are many examples in which students' interactions in groups lead to different experiences for different groups of students (e.g., Barron, 2000; Chizhik, 2001; Dembo & McAuliffe, 1987; Esmonde, 2009b). When students tend not to listen to the suggestions of their peers, they are less likely to formulate a correct solution to a problem, even if correct ideas are presented within the group (Barron, 2000, 2003). When one student is especially persistent with an idea that the group disagrees with, the group's progress can be stalled (Watson & Chick, 2011). Research on group work and cooperative learning has shifted from examining how students cooperate to learn to examining how students learn to cooperate (Good, Mulryan, & McCaslin, 1992). The emphasis here is that the social and mathematical processes in which students engage have been brought to the forefront as a key issue in studies of group work and collaboration. The nature of students' interactions in groups seems to be critical to understanding how group work may actually support students to learn mathematics.

The processes by which students achieve collaborative efforts in computer settings make these settings somewhat unique. In the culmination of a multi-year project studying students' work in pairs or groups with computers in mathematics classes, Hoyles, Healy, and Pozzi (1994) identified some of the characteristics of successful and unsuccessful groups. Similarly to more typical group work settings, successful groups and pairs emerged when students developed a shared sense of responsibility for completing the task. In addition, there were two factors related

to students' use of the computer software that were important for positive group outcomes. First, all students needed to have a sense of ownership over the computer constructions. This means that, even if multiple students were working together at one computer, all students needed to contribute to what they created on the computer screen. Unsuccessful group work resulted when one student, termed the "director", dominated the use of the computer and assumed a higher status than the rest of the students in the group. Second, students needed to engage in some discussion about the outputs resulting from their work on the computer. The activity of stepping back from the computer allowed students to focus on the mathematical meaning of their work. When all of students' work was focused on the computer, and there was no discussion away from the computer, students' group or pair work at computers was not as successful. This finding highlights how important is it for students to talk with each other about mathematics, away from the computer, even in the context of a computer environment.

There has been some suggestion that the gender composition of pairings correlate with pair behavior and learning at the computer, with boys in single sex pairs or mixed sex pairs performing better overall (Barbieri & Light, 1992; Hughes, Brackenridge, Bibby, & Greenhough, 1988). However other research has suggested that factors such as gender, age, and prior computing experience are not reliable predictors of students' success in pair work at the computer (Webb & Lewis, 1988). Much of this research has been conducted with groups of students in early or middle grades, and these studies have varied with regards to the prevalence of mathematics concepts in students' work. In general, however, it seems that no individual factor determines the quality of group or pair work at the computer. The processes through which students engage in work at the computer do have implications for their success. For group and pair work at the computer to promote mathematical learning, students need to share

responsibility over the task and computer, and students need to take time to reflect on the mathematical ideas beyond establishing a solution on the computer.

Theoretical Framework

To frame this study, I first use a *sociocultural* (Lave & Wenger, 1991; Vygotsky, 1978) perspective to define what it means for students to learn mathematics during pair work at the computer. I use *positioning theory* (Harré & van Langenhove, 1999) to understand how students assign positions to themselves and their peers through interactions during group or pair work. I assume that a goal of all group and pair work is that students will collaborate to learn mathematics, and I draw on the construct of *productive struggle* (Hiebert & Grouws, 2007) to consider how students' acts of positioning can support collaborative efforts. Finally, I use Systemic Functional Linguistics (Halliday & Matthiessen, 2004) to describe how students enact positions through their speech, and to examine the interpersonal and mathematical aspects of that positioning.

A Sociocultural Perspective on Learning

Current research on group work in mathematics is largely driven by a theory of learning deemed *sociocultural* or *situative* (see, e.g., Lave & Wenger, 1991; Rogoff, 1990; Saxe, 1991; Wenger, 1998). Sociocultural theory builds largely from the work of Vygotsky (e.g., 1978), who argued that social activity is of primary importance, while individual thinking is secondary and is derived from social activity¹⁵. This makes sociocultural theory distinct from cognitive constructivism, which prioritizes the individual as the primary unit of analysis (Cobb, 1994). Vygotsky (1978) argued that individual thought is derived from socially constructed ways of

¹⁵ Leontiev (e.g., 1981) also contributed to the perspective that social activity is a primary unit of analysis, although Leontiev and Vygotsky diverged on the question of whether individual thought stems from activity or from interactions with others.

communicating, which are practiced through interactions with others. From a sociocultural perspective, social processes have primacy over individual processes.

Sociocultural theory, where priority is placed on social activity, redefines what it means to learn. Sociocultural theory posits that *learning* is a process of enculturation into the activities of a community (Wertsch, 1998). Enculturation is itself a process of assuming the values and practices of a community. This emphasizes that learning is a social process, because it requires participating in the practices of a community. Different communities have different values and practices. For example, communities of tailors have a set of practices that include designing, sewing, and pressing garments (Lave, 1988). Street vendors have practices for selling and making change (Carraher et al., 1985; Saxe, 1991), and groups of friends have practices for playing and keeping score in basketball games (Nasir, 2000). Mathematicians have certain practices by which they engage in the work of doing mathematics (Resnick, 1989; Schoenfeld, 1992). The idea that learning is *situative* reflects the different practices of different communities. The process of learning is situated within different communities.

How enculturation into a community happens has been a topic of examination. Lave and Wenger (1991) describe this process as increasing one's participation in a community of practice, which differs according to different communities. Using the example of a community of tailors, Lave (1988) outlined how novice tailors first engage in small ways in the practices of the community, for example pressing garments and sewing buttons. Over time, novices increase their participation in the practices of tailoring, eventually designing the garments from the beginning. Lave and Wenger's (1991) idea of learning as *participation* is relevant for an examination of students' learning in mathematics classrooms. Analogously to tailoring, mathematics can be thought of as "a set of practices of inquiry and sense-making that include

communication, questioning, understanding, and reasoning,” and therefore, “learning mathematics is marked by increasing participation in an extended range of such practices” (Greeno & MMAP, 1997, p. 104). Schoenfeld (1992) similarly defined learning mathematics as a process of enculturation into the mathematics community. Increasing one’s participation in mathematical practices is the process by which students become enculturated into the mathematics community. Therefore, learning can be thought of as increased participation in mathematical practices such as communicating, questioning, and reasoning about mathematics.

I use this definition of learning as a complement to the constructivist view of learning as shifting between conceptions. Combining sociocultural and constructivist perspectives of learning has been deemed problematic by some (Confrey, 1995; Lerman, 1996). However, in mathematics education research, combining sociocultural with constructivist theories of learning has been recognized as a useful way to examine how social activities give rise to problematic situations that create opportunities for learning (Cobb, Boufi, McClain, & Whitenack, 1997; Cobb, Wood, & Yackel, 1991; Ernest, 1994; Whitenack, Knipping, & Novinger, 2001). Sociocultural theory provides a way to understand the processes by which students learn mathematics through group work, as well as the outcomes of group work activities.

There are theoretical justifications, based on this sociocultural perspective, for giving specific attention to students’ interactions during group and pair work. Although students can learn mathematics through whole-class and individual settings, the perspective that students learn mathematics through classroom social processes places a large value on students’ interactions with one another (Cobb, 1994; Cobb et al., 1991; Steffe & Tzur, 1994). Social interactions between peers within a group give rise to students’ learning opportunities in ways unique from teacher-student interactions. For example, when students work together, they have opportunities

to develop a shared appreciation for what each other finds problematic (Cobb et al., 1991). When students devise different solution strategies, they will be pressed to make their reasoning more explicit, and through that process may be able to refine their reasoning. At even a more basic level, students' abilities to cooperate within their groups—situating a piece of paper so that everyone can see the same thing at the same time, or sharing a computer keyboard—has implications for the nature of the mathematical activity that students will achieve. This basic level of interactions determines what opportunities students will have to engage in more sophisticated mathematical practices. The social problem of achieving basic cooperation has priority in that it is a necessary condition for students to learn mathematics through group work (Cobb et al., 1991). The mathematical and the social aspects of a classroom environment are inextricably linked, and the way mathematical understanding is established between students depends largely on the nature of interactions within a classroom.

Students' interactions with peers during group and pair work give rise to problematic situations, allow students to communicate with one another about mathematics, and create opportunities for students to refine their reasoning. All of these activities reflect ways by which students can increase their participation in the practices of mathematics. However, it cannot be assumed that students will necessarily engage in these practices, or that all students will have equal opportunity to engage in such practices. I use the lens of positioning theory to consider how students enact positions towards their peers during pair work, with the goal of understanding when and how students can create opportunities for learning through their interactions.

Positioning Theory as a Framework for Studying Students' Interactions

Positioning theory offers a framework to understand how individuals enact different positions through their interactions with one another. *Positioning*, according to van Langenhove and Harré (1999), refers to the ways people use action and speech to assign parts to speakers, with the purpose of giving structure to a person's actions. Positioning highlights the way that speakers assign and reassign parts on a moment-by-moment basis. Positioning is particularly important for thinking about pair work because the ways that students position themselves and one another have implications for the opportunities that they will have to participate in mathematical activity. For example, a student who is consistently positioned as the smartest student in a mathematics class may have more opportunities to control a discussion than a student who is positioned as less smart (Cohen, 1994a). This type of pattern is likely to impact not only what mathematics is explored and discussed during group work, but also how different students, specifically those who are not positioned as the expert, come to engage in the discussions of that mathematics.

Although students constantly position themselves in interactions, this does not imply that all students have the opportunity to position themselves in the ways that they want. In social settings, and particularly in small groups, students who hold more power have an advantage over those with less power. Drawing on Foucauldian ideas of the nature of power, Gutiérrez (2013) argued that power is not owned by any individual but rather power is “circulated through discourses.” Those individuals who participate in the dominant discourses of a particular context achieve a certain status and thus are in a position to assert force over those who do not. Practically speaking, power in mathematics education, as it is played out through social interactions in the classroom, refers to the status of those who most easily assimilate into the

dominant discourse of the classroom. This point is especially salient in thinking about the opportunities that students have to position themselves. Those students who hold status in a classroom as being “good” at mathematics are more likely to be well versed in the dominant discourse of the mathematics classroom (Gutiérrez, 2013; Zevenbergen, 2000). Similarly, students who have grown up speaking English at home, or students who have been on an honors track throughout their mathematics education, have more access to the dominant discourse of mathematics, and therefore are better enabled to position themselves in the ways that they want (e.g., Chizhik, 2001; Moschkovich, 1999). Research has shown that students who come to school already participating in the mainstream practices of school tend to be privileged when participating in classroom conversations (Lubienski, 2000; Nasir, Rosebery, Warren, & Lee, 2006). Students’ preexisting, socially constructed identities (e.g., intersections of gender, race, prior achievement levels) impact the opportunities that students have to position themselves in positive ways.

Empirical studies of positioning practices in mathematics classrooms have been carried out on multiple levels. Studies of curricular materials have found that the language used in textbooks often positions students as under the authority of the teacher and the book (Herbel-Eisenmann, 2007; Herbel-Eisenmann & Wagner, 2007). This observation came specifically in the context of reform-minded curricular materials that were designed to give students more agency over their mathematical learning. Similarly, traditional classroom settings often prescribe a certain pattern of positioning between teachers and students (Herbel-Eisenmann & Wagner, 2010). During typical classroom activities, such as the work of a teacher leading a review, students tend to have little agency to control the class discussion, even when they have opportunities to ask and answer questions (González & DeJarnette, 2012). Typical classroom

structures, where the teacher is seen as the mathematical authority and expert, and students are expected to acquire knowledge from the teacher, reinforces students' and teachers' positioning practices in this way. Studies of group work and pair work in mathematics classrooms illustrate that, when students work together and not under the direct authority of the teacher, students distribute authority between themselves (e.g., Esmonde, 2009b; Hoyles et al., 2004).

Positioning has implications for students' mathematical learning to the degree that it impacts how they are able to identify with the discipline of mathematics. A useful definition of identity is a dynamic view of self, as it becomes manifest through social interaction, through the ways that people position themselves and are positioned by others (Davies & Harré, 1990). Learning in mathematics classes refers to more than students becoming socialized into mathematical practices. Learning also requires changes to how students see themselves with respect to mathematics (Esmonde, 2009a). For students to increasingly participate in mathematical practices such as reasoning and communicating, they should position themselves, and be positioned, as capable of engaging in those practices. When considering students' opportunities to learn mathematics, it is as important to take into consideration their access to positional identities as knowers and doers of mathematics as it is to consider their access to mathematical content (Gresalfi & Cobb, 2006). The ways students engage with mathematics content will impact their identities. At the same time a student's view of him or herself is also likely to impact the ways in which he or she participates in mathematics.

Recognizing that power is not owned by any individual allows for an appreciation of how students have potential to redefine power relationships (Gutiérrez, 2013). Positioning may reinforce power relations, as in the case of granting authority to the high-status students in the group, or positioning can be used as a tool to re-align the power dynamics within a group.

Walshaw (2001) found that in classroom contexts where femininity was devalued, female mathematics students drew on other discourses of which they were a part in order to gain access to more power. Similarly, Esmonde and Langer-Osuna (2013) found that an African American female student drew on her status within the social structures of the classroom to assume more power within the mathematical discussion. Students draw on different aspects of their identities and contexts when positioning themselves towards their peers. There are multiple factors that contribute to students' positioning in mathematics classrooms. Students are positioned according to socially constructed categories, and also students position themselves and are positioned through moment-to-moment acts of positioning (Esmonde, 2009a; Hodge, 2006).

Positioning in Students' Interactions

Students' moment-to-moment positioning practices towards one another have become a central focus of attention of much research in recent years. Esmonde (2009b) identified three different positions that students regularly took up—experts, novices, and facilitators—when working on different group oriented activities. Esmonde found that the nature of students' collaboration varied with regards to the positions of the members of the group, as well as the structure of the activity. For example, during group quiz activities, groups with a clearly established expert rarely established any collaborative practices. Instead, the expert of the group adopted a practice of “helping” other students in the group by telling the answers. However, in group presentation preparations, even groups with established experts were more likely to maintain a level of collective problem solving. The findings of this study suggest that students' positioning practices impact who has what opportunity to contribute mathematical ideas. Moreover, this study suggests that students' acts of positioning have different implications according to the type of activity in which they are engaged. A study of middle school students

using a computer-programming environment at the introduction of a geometry unit has suggested that introducing a technology environment can create opportunities for students to reposition themselves among their peers (Fields & Enyedy, 2013). In this study, two students who had not previously been positioned as leaders in their class—but who had gained experience programming through an after school club—had an opportunity to be experts relative to their peers when using the programming. Overall, students often change positions, and contribute differently to group work, based on the type of activity in which they are engaged.

In previous work, I have used Esmonde's (2009b) notions of expert and novice students to examine how students' positioning practices supported them to establish the resources, operations, and products necessary for achieving a solution to a problem in an Algebra 2 class (DeJarnette & González, 2013, April). In that work, we found that the ways in which students initiated positions within their group had implications for how much conversation the group devoted to the different components of the task at hand. For example, in one group, one student was positioned as the expert in the group by herself and by her peers. The members of the group asked the expert questions about the solution to the problem, and they accepted the answers she provided. In other groups, different students positioned themselves as experts at different moments during the problem solving process, without necessarily being seen as the expert by the other members of the group. In the groups with no single expert, students were more likely to challenge one another's positioning, thereby creating opportunities to make the resources and operations necessary for completing the task more explicit. Students' acts of positioning, and moreover their challenges towards others' acts of positioning, provided a way for the students to collectively engage in the different aspects of the task.

A common theme among the above studies is that students' positioning practices within small group contexts influence what mathematical ideas are brought up and how those ideas are discussed. Students can also use acts of positioning in ways that silence one or more members of a group. In a study of middle school students working in groups to construct a box of maximum volume given certain constraints, Kostopoulos (2013) found that one student, Mitchell, not only had his ideas ignored by his group members but also was effectively silenced in his idea about how to construct the box. Although Mitchell had one of the highest achievement scores in the entire class of students (and his strategy was the closest of those in the group to being correct), he was positioned as one of the "low achievers" by the other students in his group. At times Mitchell made moves to position himself as more competent within his group, but his group members did not reciprocate this positioning. By the end of the study, Mitchell had dismissed his own strategy for creating a box.

There is a perspective that group work activities may rely on the assumption that some students have more expertise than their peers (Webel, 2010), and therefore group work may provoke students to divide themselves according to who are experts or novices. For example, training students to give better explanations to their peers (e.g., Fuchs et al., 1997; Webb & Farivar, 1994) relies on the assumption that a single student will be giving an explanation to another student. In a classroom devoted to inquiry and group work, Goos (2004) found that, in the absence of teacher-directed learning, some students assumed the authoritative position of the teacher, scaffolding the students they identified as their less-able peers. Webel (2010) suggested that students' collaborative behaviors may be more productive when collaboration is seen to serve the purpose of *co-constructing* ideas, rather than the purpose of helping all students learn mathematical content. A vision of group work as *knowledge building* is more reflective of the

work that occurs between mathematicians within research institutions (Scardamalia & Bereiter, 2006), and it may result in students positioning themselves more symmetrically while generating knowledge within a group.

Research on students' positioning practices suggests that there are multiple factors at play as students engage in moment-to-moment positioning in their group. Socially constructed categories related to identity, including gender, race, and socioeconomic status, impact the opportunities students have to position themselves in the ways they want. At the same time, acts of positioning occur through dynamic interactions between individuals and cannot be assumed to be entirely a function of categorical identity distinctions. Many factors contribute to students' positioning practices, and the ways that students position themselves are intertwined with how different students engage in mathematical activity. A detailed examination of how students enact positions through their talk, while solving a problem in pairs, can offer insight into how different patterns of positioning can support students to learn mathematics through collaboration with peers.

Collaboration and Productive Struggle

Positioning theory offers a way to understand how students interact during group and pair work. To understand how students' positioning practices support mathematical learning, I consider students' positioning to the extent that those practices either support or inhibit collaboration between students. Ideally, group and pair work in mathematics classrooms should support students to collaborate to solve problems and make sense of mathematics. When students *collaborate*, they jointly contribute to formulating and solving problems, and through that process making sense of mathematical ideas (Staples, 2007). As a result, students who learn mathematics through collaboration should have shared ownership of the mathematical ideas that

emerge. Collaboration is not automatically a result of group or pair work, and collaboration is distinct even from *coordination*. Students can coordinate with one another to complete a task, sharing answers with one another or making independent contributions to a final product. This distinction is important, because in my examination of students during pair work I view collaboration as a form of mathematical learning. Considering that learning is a process of increasing one's participation in the practices of sense-making, questioning, explaining, and reasoning, collaboration with peers is one activity through which students learn mathematics in the classroom.

An assumption I make, based on Staples's (2007) definition of collaboration, is that for students to collaborate, they must engage with a problem for which they do not immediately know the solution. Hiebert and Grouws (2007) identified an activity of *productive struggle* to mean that "students expend effort to make sense of mathematics, to figure something out that is not immediately apparent" (p. 387). Students' struggle should be productive, because they should work on problems that are within reach, and they should use mathematical ideas that are not yet well formed but are tangible (Hiebert et al., 1996). Struggle does not refer to feelings of frustration or despair, but instead it refers to a process of active work to make sense of a situation. In the work of Hiebert and Grouws, productive struggle does not have to be an activity occurring between students during group work. An individual can struggle to make sense of mathematical ideas. In this study, I am most interested in what I term *collaborative productive struggle*, or the process by which students engage in collaboration to make sense of mathematics. With the construct of collaborative productive struggle, I seek to establish a link between students' positioning practices and the mathematical practices they engage in during pair work. I

study students' discourse as a way to identify when they engage in collaborative productive struggle.

Using Linguistics to Examine Students' Positioning

Systemic Functional Linguistics, or SFL (Halliday, 1984, 1994; Halliday & Matthiessen, 2004) provides a way to explicitly and quantifiably describe dialogic interactions, as well as a way to interpret the structure of dialogue as a reflection of interpersonal relations (Eggins & Slade, 1997, p. 180). Halliday (1984) suggested that conversation is a process of exchange involving two variables: some commodity to be exchanged (either information or goods and services), and the roles that the interactants take on (either giving or demanding). Combining the two variables above defines the four basic speech functions, in other words the four basic types of moves that interactants can make in order to initiate a dialogue. A person can make a statement, ask a question, offer goods or services, or demand goods or services (Halliday, 1994). An implication of the four basic speech functions is that, in initiating an exchange, an interactant positions not only him or herself but also the other party engaging in the exchange:

When the speaker takes on a role of giving or demanding, by the same token he assigns a complementary role to the person he is addressing. If I am giving, you are called on to accept; if I am demanding, you are called on to give. (Halliday, 1984, p. 12)

The four speech functions, giving or demanding information or services, makes salient two of the important functions of language in the theory of SFL¹⁶. Language serves an *interpersonal* function, which focuses on the ways relationships are construed through the language. At the

¹⁶ In SFL there are three metafunctions of language, the *interpersonal*, the *textual*, and the *ideational*. The ideational metafunction focuses on the content being communicated (Halliday & Martin, 1993; Halliday & Matthiessen, 2004). All metafunctions of language occur in every interaction, but for this study I pay specific attention to the interpersonal and the textual functions.

same time, language serves a *textual* function, which focuses on how language is organized. With the four speech functions, the activities of giving versus demanding reflect the interpersonal nature of a communication. The textual metafunction makes apparent how students talk about and refer back to ideas over the course of a lesson. I use SFL because, through the interpersonal and textual metafunctions, SFL identifies how individuals organize and keep track of the content of their talk, as well as how speech roles position both the speaker and the respondent during such interactions.

Drawing on the work of Halliday, others have developed the system of Negotiation to give a finer description of the moves that speakers may use to serve the interpersonal metafunction of language (Berry, 1981; Eggins & Slade, 1997; Love & Suherdi, 1996; Martin, 1992; Ventola, 1987). In the system of Negotiation, there are two types of exchanges: knowledge exchanges and action exchanges. Participants in an exchange can be distinguished between who is the *primary* knower or actor (denoted by K1 or A1, respectively) and who is the *secondary* knower or actor (denoted K2 or A2, respectively). A primary knower is a person who has some information. In the context of students working together in a group or pair, the primary knower in a given exchange is the person to whom another is deferring to provide the answer. For example, Student A may ask, “What is the cosine of 0?” And Student B may respond, “The cosine of 0 is 1.” In this example, Student A asked a question, thereby making a request for some information. Student B provided that information. Student A initiated an exchange by performing a K2 move, and Student B responded with a K1 move.

An action exchange is identified according to who is the primary actor (A1) and who is the secondary actor (A2). The primary actor is the individual who actually performs an action or provides a service (e.g. reading the text out loud, turning the page, performing a computation on

the calculator), and the secondary actor is the person who makes the request for such an action to be performed. For example, suppose Student A has said to Student B, “You work on the first problem.” With this statement, Student A has made a request to Student B to perform the action of working on a particular problem. In making the request, Student A performed an A2 move. Assuming Student B complied, he would have performed an A1 move.

The moves discussed above, primary or secondary knowers (K1, K2) or actors (A1, A2), are known as *synoptic moves*, regular and predictable moves occurring in exchanges of information and action. However, few exchanges are as predictable as requiring only one K2 move and one K1 move. There are a variety of other ways in which interactants make themselves understood in conversation, and one of these ways is by using multiple moves in order to explain an idea. When a speaker performs a sequence of moves that are all the same status, they create what is called a *move complex* (Martin & Rose, 2007; Ventola, 1987). To distinguish between distinct moves, and to establish connections between moves within a move complex, there are three possible logical relations: elaboration, extension, and enhancement (Halliday, 1994; Halliday & Matthiessen, 2004; Love & Suherdi, 1996; Ventola, 1987). An elaboration (=) is a move that is a restatement or rephrasing of a move that has already been made. An extension (+) is a move that adds some information to a previous move. And an enhancement (x) acts as a qualifier to a previous move. Move complexes serve as one way for students interacting during group work to make their ideas explicit beyond an initial statement or question. Move complexes serve multiple functions in conversations between members of a pair or group. Move complexes allow one student to contribute multiple ideas in a single turn, and they allow for more mathematical ideas to be brought up for discussion.

Beyond these initial distinctions, the system of Negotiation provides a finer description of the conversational moves that speakers make, for example delaying a response, clarifying an initiation, rephrasing a question, or challenging a statement (Love & Suherdi, 1996; Martin, 1992; Ventola, 1987). Dynamic moves are those moves which “represent the complex nature of real discourse, where messages are misunderstood, re-enforced, abandoned, clarified, or corrected” (Love & Suherdi, 1996). Ventola (1987) identified three systems of dynamic moves. *Suspending* moves are used to make sure that a statement has been heard correctly: requesting confirmation (cfrq), giving confirmation (cf), backchanneling (bch), and checking (check). *Aborting* moves challenge the validity of a prior statement, attempting to extricate one of the participants from an exchange: challenge (ch), and response to challenge (rch). *Elucidating* moves attempt to clarify an initiation before a response is given: clarification (clfy), and response to clarification (rcly). Finally, Love and Suherdi (1996) identified a list of *sustaining* moves that grew out of a study specifically of classroom discourse. Sustaining moves act as an attempt to sustain an initiation in order to make sure that it has been heard and understood correctly: repeating (rp), rephrasing (rph), clue (clue), correction (corr), irrelevant response (irr), and no response (ro). Taken together as a system, synoptic moves, move complexes, and dynamic moves give a detailed picture of the interactions that occur in classroom conversation. For a complete list of Negotiation moves and their codes, see Appendix G.

Connecting Students’ Conversations to Positions in Group Work

Applying the system of Negotiation to the interactions between group members provides a way to analyze how students position themselves relative to one another during group work. To make sense of this positioning, one can consider the correspondence between moves from the system of Negotiation and the roles that Esmonde (2009b) identified as surfacing during

cooperative group work: expert, novice, and facilitator. In previous work, I used operational definitions of expert, novice, and facilitator to identify the positions that students took up during group work (DeJarnette & González, 2013, April). Table 4.1 provides an overview of these operational definitions. In a group of students, an *expert* was a student who was deferred to mathematically, and who was given the authority to determine whether an answer or idea is correct. Within the system of Negotiation, an individual would assume a position of expert by performing a K1 move. The *novice* in a group would be an individual who deferred to an expert, or the person whose ideas or solutions were passed over in favor of another's. In a Negotiation exchange, a person performing the K2 move would be the novice. Individuals might use K2 moves for two different reasons: either to ask an explicit question (e.g., "how do we do that?"), or to suggest an idea to be evaluated (e.g., "so is the cosine of 0 equal to 1?"). In either use of a K2 move, the novice defers to another member of the group to perform the K1 move.

Table 4.1
Group Positions From the System of Negotiation (DeJarnette & González, under review)

Role	Esmonde's (2009b) Definition	Linguistic Markers
Expert	Frequently deferred to mathematically; granted authority to determine whether her own and others' work is correct. An expert positions herself as such, and also is positioned as such by peers.	Performs K1 moves in the majority of exchanges. Resolves challenges or requests for confirmation or clarification made by the novice.
Novice	Defers to an expert, positioning herself as less competent. Often instructed by others, though she may sometimes question the experts advice.	Performs K2 moves in majority of exchanges. May perform dynamic moves to question, challenge, or suspend an exchange.
Facilitator	Orchestrates group activity, and fosters participation of group members. Makes sure that all group members participate in some way, or, ideally, actively encourages group members to participate in joint problem solving.	Performs dA1 or A2 moves. Uses these moves with the purpose of offering to perform an action

Finally, we identified the facilitator in a group as the person who orchestrated the group activity. In an exchange, an individual might facilitate the activity by offering to do something for the group or coordinating the actions of others. In the system of Negotiation, a facilitator would be the person performing dA1 moves, to offer to do something for the group, or A2 moves, to coordinate or direct what others would do. In our analysis we found that students' use of action moves during group work, and specifically A2 moves, served a variety of purposes towards group work, including directing the members of the group to wait or slow down and telling other members to pay attention to a specific part of the problem. Although students enacted positions of experts and novices in the ways that we expected during their work in groups, students did not facilitate group work in the way that we expected prior to our analysis.

Much of the prior research about students' positioning practices towards their peers has been conducted in contexts of students working together in groups of three or more. In this study students worked together in pairs, and therefore I gave careful consideration of how research on students' positioning practices could inform this work. Specifically, one could expect that in a pair of two students, students would be limited in their opportunities to position themselves or each other. For example, if one student consistently positions herself as the expert, then the other student in the pair is compelled to be positioned as the novice. In any given exchange between two students, there can be at most two positions that students take up. In addition, students' use of action moves could be indicative of a variety of different positioning practices. Drawing on the research on the variety of positions that students take up towards their peers, I expected that students in this study would take up multiple positions even during pair work. However, I expected that students' positioning would look somewhat different in this context than it has in typical group work settings.

Data and Methods

The data for this study are from an investigation of the nature of students' mathematical learning through work on an open-ended problem with the use of a computer-programming environment. Students participating in the study were in three different sections of Algebra 2 taught by Ms. Alexander¹⁷ at Grove High School. Grove High School is a school of approximately 1,000 students, where around 30% of students are Latino/a and 30% of students qualify for free or reduced-price lunch. The students in Ms. Alexander's Algebra 2 classes were on the regular track (i.e., non-honors) of mathematics classes at the school. Most of the students in the study were in 11th grade, with a small number of students in 12th grade. This study focuses on a lesson that Ms. Alexander taught at the conclusion of the unit on sine and cosine functions, a lesson about using sine and cosine functions to represent real world phenomena. The lesson was an *instructional experiment* (Herbst, 2006). Instead of teaching the typical lesson out of the textbook, Ms. Alexander introduced students to the "Ferris wheel problem," for which students were to use a computer-programming environment called Etoys to represent their height off the ground as a function of time while riding a Ferris wheel.

Students participating in the study worked in pairs, with two exceptions (Table 4.2). My decision to have students work together in pairs at the computer is based on research on the efficacy of *pair programming* in computer science education (Williams, Wiebe, Yang, Ferzli, & Miller, 2002). Pair programming is a style of programming in which two programmers work side by side at one computer. Pair programming is designed to promote collaboration between learners, where the two learners within a pair work jointly on almost all parts of a task. I expected that having students work in pairs, rather than larger groups, would allow for both

¹⁷ I use pseudonyms for all participants and institutions.

students to be involved in the work at the computer. Ms. Alexander did not instruct students on how they should interact or how they should share responsibility of the computer. However, Ms. Alexander did tell her students that she was treating the lesson as a “rich task,” which meant that students were expected to talk with one another and work towards a solution to the problem, even if they were not sure of the solution. Ms. Alexander’s students had experience working on rich tasks regularly in Algebra 2. Although they did not have specific norms in place for how to interact, students knew that they were expected to share ideas with their partners and rely on one another to solve the problem.

Table 4.2
Students Participating in the Study of the Etoys Lesson

Carson Abbey	Jalisa Zach	Gia Courtney	Hannah Dayana	Shane Maya	Lucas Elizabeth Andy
(Bailey) Aubrey	Cara Maggie	Mike Jessa	Mitchell Reese	Tori Sean (Aubrey)	

Note: Bailey was only present on the first day of the lesson. On the second day of the lesson, Aubrey worked with Tori and Sean.

There were two groups of three in the study, one of which was the group of Lucas, Elizabeth, and Andy. The choice to allow for this group of three in the study was because, in one class period, an odd number of students signed up to participate in the study. Had a participant of the study been paired with a non-participant, I would not have been able to collect video or audio recordings of that pair of students. To gain information from a larger number of students overall in the study, I instead allowed for one group of three students to work together over the 2-day lesson. In addition, although Bailey and Aubrey worked together on day 1 of the lesson, Bailey was absent on day 2 of the lesson. On day 2, Aubrey joined Tori and Sean’s group.

Throughout this study, I will refer to the pairs of participating students in the study, though I include the groups of three in that reference. When I have findings specific to, or unique from, the groups of three students, I will make that explicit. A brief overview of the mathematical concepts at play in the Ferris wheel problem, as well as the possible solution strategies, will provide necessary background before outlining the data sources and methods for analysis.

An Overview and Solution to the Ferris Wheel Problem

To solve the Ferris wheel problem (Figure 4.1) students had to construct a function that would represent their height off the ground as a function of time while riding a Ferris wheel. There were two slightly different versions of the Ferris wheel problem. In one version, students were given that the diameter of the Ferris wheel was 130 meters, and it took 30 minutes to make one complete revolution. In the second version of the problem, students were given the diameter of the Ferris wheel to be 140 meters, and it took 40 minutes to complete one revolution. Both versions of the problem could be represented with either a sine or cosine function, with minor modifications to the parameters of the function.

One of the most famous Ferris wheels in the world is the London Eye in London, England. Assume that the London Eye has a diameter of 130 meters, and the lowest point on the Ferris wheel is 5 meters above the Thames River. It takes 30 minutes to make one complete revolution. You and your partner are going to ride the Ferris wheel. You get on the Ferris wheel at the very lowest point.

Figure 4.1. The Ferris wheel context.

Students needed to recognize that the trip around the Ferris wheel could be represented by a sine or cosine function with the appropriate parameters. After recognizing the appropriateness of a sinusoidal function, students needed to construct the function to represent the context according to the given diameter and period of revolution of the Ferris wheel. There were two main steps towards working to a solution to the problem. First, students would use

Etoys to plot points representing their height off the ground at various moments (Figure 4.2).

Based on this representation, students should have recognized the periodic nature of the graph and decided to use a sine or cosine function to represent the scenario.

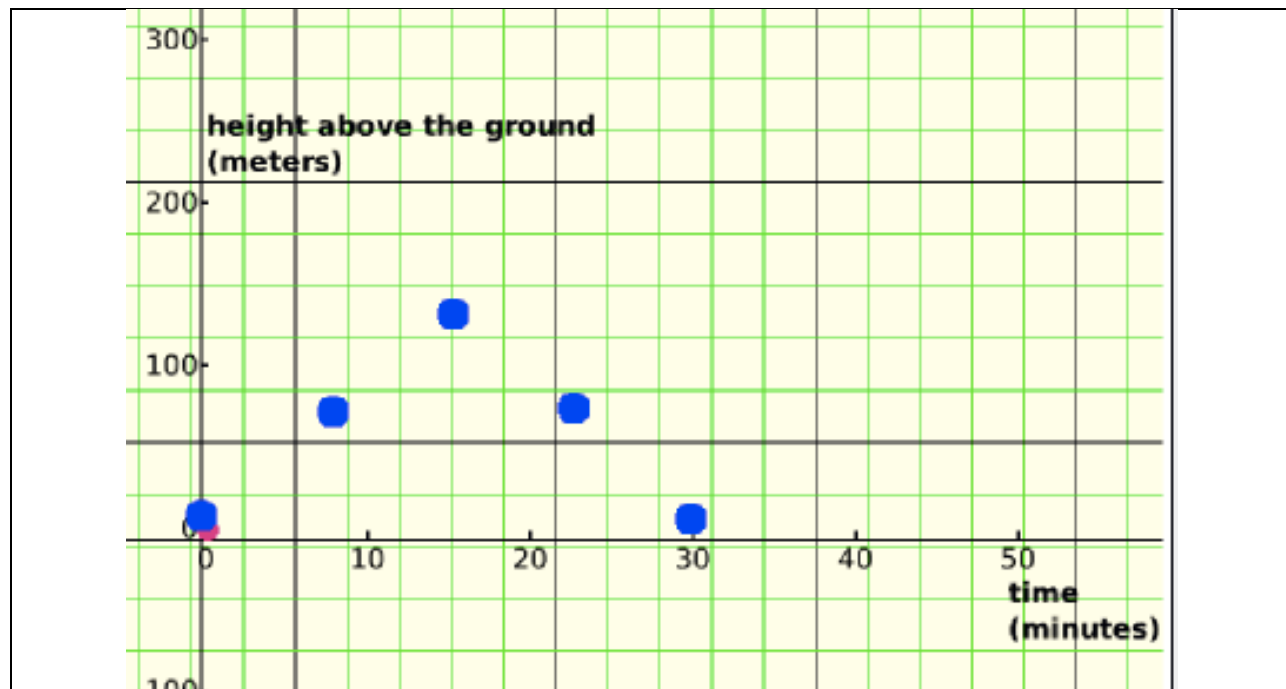


Figure 4.2. Plotting points to represent one's height off the ground while riding the Ferris wheel.

Once students decided to use a sine or cosine function, they needed to use Etoys to write a “script” that would represent their height off the ground as a function of time (Figure 4.3). A script in Etoys is a short program. Students in Ms. Alexander’s class had not used Etoys prior to the day of the Etoys lesson. I designed the Etoys file that students would use, in collaboration with another researcher who has used Etoys in settings, so that students would have enough resources to begin working on the problem as they learned to use Etoys (González, 2013, personal communication; Lundsgaard, 2013, personal communication; Pitt, 2013, personal communication). In the Etoys file, I provided students with an example of a script that would plot a quadratic function, so that students could refer to the example as they constructed their own scripts. Constructing such a script required students to write a rule that would assign the

appropriate value to the y -coordinate of the “plotter” (a programmable object) based on its current x -value. Then, students needed to program the plotter to make a stamp, and then increase its x -value by some fixed amount before running the script again. Running the script would create a graph that went through the collection of discrete points students had previously plotted.

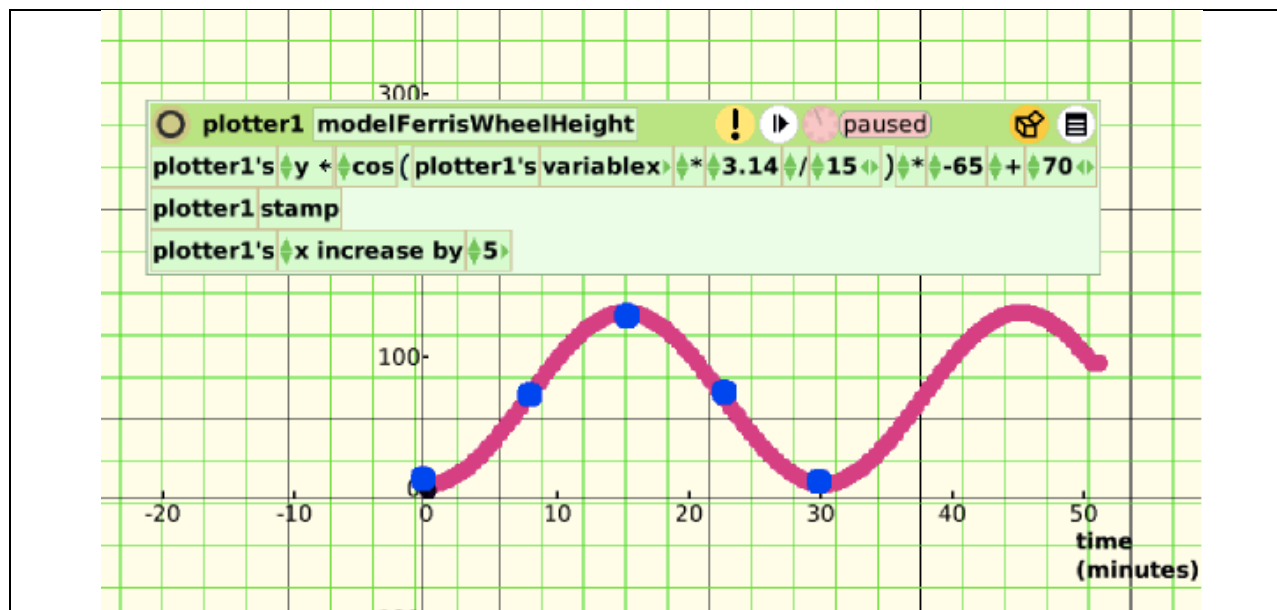


Figure 4.3. A script to represent one’s height off the ground while riding the Ferris wheel.

The crux of students’ mathematical work was to write the rule to assign a value to the y -coordinate of the plotter. Students could have used either a sine function or a cosine function to represent the scenario. To solve the version of the problem in Figure 4.1, students could have

used the functions $f(x) = -65 \cos\left(\frac{2\pi}{30}x\right) + 70$, $f(x) = 65 \sin\left(\frac{2\pi}{30}x - \frac{\pi}{2}\right) + 70$, or

$f(x) = 65 \sin\left(\frac{2\pi}{30}x + \frac{3\pi}{2}\right) + 70$. Functions to represent the alternate version of the problem (with

diameter 140 and revolution time 40 minutes) would be $f(x) = -70 \cos\left(\frac{2\pi}{40}x\right) + 75$ or

$70 \sin\left(\frac{2\pi}{40}x - \frac{\pi}{2}\right) + 75$. To write the appropriate function for either version of the problem,

students needed to account for the amplitude, period, and horizontal or vertical shift of the sine or cosine function.

To account for the parameters of the function, students would need to use the information about the diameter of the Ferris wheel and its height off the ground. In the case of the 130 meter Ferris wheel, one's height off the ground would range from 5 meters (at the very bottom of the ride) to 135 meters (at the very top of the Ferris wheel). Represented by a sinusoidal function, this would mean that the distance from the midline to the top or bottom of the curve would be 65 meters. Therefore the coefficient of the function would be 65 or -65, depending on whether the student used a sine function or cosine function to represent the situation.

A sinusoidal function with a coefficient of 65, and no other parameter adjustments, would range from a minimum value of -65 to a maximum value of 65. To account for the fact that the heights would need to range from a minimum of 5 to a maximum of 135, students needed to add a vertical shift to their functions. By adding 70 to the end of the sinusoidal function, the graphical representation of the function would shift vertically by 70, to give the appropriate maximum and minimum values. If students chose to use a sine function (instead of a cosine function) to represent the situation, they would also need to subtract $\frac{\pi}{2}$ inside the sine function to give a horizontal shift.

To account for the final component of the sinusoidal function, the period, students would need to multiply the independent variable by some number to stretch or compress the graph of the function horizontally. Given a sine or cosine function with period P , the coefficient of the independent variable inside the sinusoidal function must be equal to b , where $P = \frac{2\pi}{b}$. Since the period of revolution for the Ferris wheel was 30 minutes, students could use the equation

$30 = \frac{2\pi}{b}$ to compute the coefficient to be $\frac{2\pi}{30}$, or $\frac{\pi}{15}$. Multiplying by the coefficient of $\frac{\pi}{15}$

inside the sinusoidal function would stretch the sine or cosine graph horizontally, so that instead of having a period of 2π (or approximately 6.28), the graph would have a period of 30.

I did not expect that students, when working on the problem, would solve for each component of the sine or cosine function independently from the others, in the way that I presented the solution above. Instead, I expected that students would pay attention to different aspects of the problem at different moments, depending on what became apparent through their experimentations with Etoys and their conversations with their partners. This expectation made my third research question, about how students' positioning practices affected their problem solving processes, especially important. Namely, I identified a need to examine how students' positioning practices were related to how students navigated the relevant mathematical concepts, procedures, and problem solving strategies they used in their work.

Parsing and Coding the Transcript

The first step of the analysis was to parse the transcript into Negotiation moves, the smallest unit after which a speaker change could occur without the transfer being seen as an interruption (Egins & Slade, 1997, p. 186). A move may be larger or smaller than a clause, depending on whether a natural break could occur between or within clauses. There are two important indicators of what constitutes a move in the system of Negotiation: (1) the independence or dependence of clauses, and (2) the prosody (rhythm or intonation) of the speaker. If a speaker were to say, for example, "If the function starts at 1, then it must be a cosine function," that would constitute a single move even though the utterance is made up of two clauses. This is because the first clause is not an independent clause. On the other hand if a speaker were to say, "The function has to be cosine because $y=1$ when $x=1$," the determination of

whether this constituted one move or two moves would depend on the rhythm of the speaker. If the speaker were to pause between the two clauses, then it could be separated as two moves, because another speaker could reasonably step in without it being seen as an interruption. If the speaker did not pause between the two clauses, I would code it as a single Negotiation move.

After parsing the transcripts into moves, I coded each move according to the system of Negotiation to identify moves and exchanges (González & DeJarnette, 2012; Love & Suherdi, 1996; Martin, 1992; Ventola, 1987). The process of parsing and coding students' conversations according to the system of Negotiation is best illustrated by an example, which I provide in Tables 4.2 and 4.3. In Table 4.2, Hannah and Dayana were looking at the virtual Ferris wheel and making a prediction about how they would represent their height off the ground over time.

The transcript conventions in Tables 4.2 and 4.3 will apply to all of the transcripts in this chapter. Parentheses around speakers' talk indicate overlapping speech between two speakers. For example, in turns 42 and 43, Hannah said, "and then it goes back down," at the same time that Dayana said, "wouldn't that be sine?" In the column for the speaker's turn, I use comments inside square brackets, [], to indicate a non-verbal action that contributes to the conversation, as in Hannah's gesture in turn 48. If a speaker paused in the middle of a turn, as in turn 45, I used brackets to indicate that pause. The colon in the word "oh:h" in turn 45 indicates that Dayana elongated the word by 1 second. For a complete record of the transcription conventions, see Appendix H.

Table 4.3

A Segment of Conversation Between Hannah and Dayana from Day 1 of the Lesson

Time	Turn #	Speaker	Move
12:55	42	Hannah	Wouldn't it be? Would it be cosine, like with a negative amplitude? And then you start here [pointing approximately to (0,5) on the plane], and then you go up, like over time, it goes up, (and then it goes back down.)
	43	Dayana	(Wouldn't that be sine?)
	44	Hannah	No because it's starting at the bottom and not at the middle.
	45	Dayana	Oh:h. [Pause 3 seconds.] It goes up then down (then up.)
	46	Hannah	(Mm hmm.)
	47	Dayana	Okay.
	48	Hannah	So it'd be cosine, but with a negative amplitude, so it would start at the bottom and then go up. Instead of [gesturing with hands a curve that starts at the top then goes down and back up].
	49	Dayana	Okay.

To code the transcript, I parsed the transcript into moves and then identified each move according to the system of Negotiation (Table 4.3). Hannah initiated this segment of conversation with Dayana by asking, “Wouldn’t it be? Wouldn’t it be cosine? Like with a negative amplitude?” I coded Hannah’s first question, “Wouldn’t it be?” as “n/m” to represent “no move” within the system of Negotiation. Although the inflection of Hannah’s voice rose, indicating that she may have intended to ask a question, the words, “wouldn’t it be?” did not identify any information or action to be exchanged. In move 42.2 (referring to move #2 within turn 42) Hannah performed a K2 move by asking Dayana if the function to represent the scenario would be a cosine function. After her initial K2 move, Hannah added some more information to her question, indicating that she was referring to a cosine function with a negative amplitude (move 42.3). The notation “K2 + (42.2)” in the coding of Hannah’s move indicates that the K2 move in 42.3 was part of a move complex, adding information to the K2 move she performed in move 42.2. After Hannah’s K2 move complex, Hannah and Dayana paused for 1 second. Since Dayana did not provide a response to Hannah’s K2 moves, I coded this pause as “ro” for

“response omitted” (move 42.3). Dayana’s omitted response concluded the exchange, which was the seventh exchange overall between Hannah and Dayana. Exchange 7 was a knowledge exchange, as opposed to an action exchange, because Hannah was asking about a piece of information. The *object of Negotiation* in the exchange was the knowledge of what function would be appropriate to represent one’s height off the ground. The exchange was not resolved, because within exchange 7 Hannah and Dayana did not establish whether or not the cosine function was the appropriate function.

Table 4.4
Coding Hannah and Dayana’s Conversation According to the System of Negotiation

Exchange #	Turn #	Speaker	Move	Code
7	42	Hannah	1. Wouldn’t it be?	n/m
			2. Would it be cosine?	K2
			3. Like with a negative amplitude?	K2 + (42.2)
			4. [Pause 1 second.]	ro
8			5. And then you start here [pointing approximately to (0,5) on the plane],	K1
			6. and then you go up, like over time, it goes up,	K1 + (42.2)
			7. (and then it goes back down.)	K1 + (42.2)
	43	Dayana	1. (Wouldn’t that be sine?)	ch
	44	Hannah	1. No because it’s starting at the bottom and not at the middle.	rch
	45	Dayana	1. Oh:h.	K2f
			2. [Pause 3 seconds.]	n/m
9	46	Hannah	3. It goes up then down (then up.)	K1
			1. (Mm hmm.)	K2f
8	47	Dayana	1. Okay.	K2f = (45.1)
10	48	Hannah	1. So it’d be cosine, but with a negative amplitude,	K1
			2. so it would start at the bottom and then go up.	K1 = (48.1)
			3. Instead of [gesturing with hands a curve that starts at the top then goes down and back up].	K1 + (48.2)
	49	Dayana	1. Okay.	K2f

After the pause, Hannah initiated a new exchange (exchange 8), this time by performing a K1 move complex. With moves 42.5-42.7, Hannah provided information about the shape of the graph, indicating that a graph of the height would start at (0,5) and then go up and back down. As Hannah was finishing her move complex in move 42.7, Dayana performed a challenge move towards Hannah in move 43.1. With her challenge move, Dayana suggested that the appropriate function would be a sine function rather than a cosine function. In response to the challenge, Hannah appealed to the idea that cosine functions start at their maximum or minimum values, but sine functions start at their midlines (move 44.1). Dayana accepted that response and performed two follow up moves, labeled “K2f” and “K2f=”, to Hannah’s sequence of K1 moves.

Exchange 9 provides a case of an exchange that is nested within another exchange¹⁸. Exchange 9 occurred within exchange 8, in between Dayana’s two follow up moves. Dayana initiated exchange 9 with a new K1 move, offering information about the shape of a graph. Based on the interaction in exchange 8, Dayana may have been referring to either a sine graph or a cosine graph when she said, “it goes down and then back up.” The important point, in terms of identifying her positioning, was that Dayana offered a statement of information, and Hannah followed up in agreement. It seems that exchange 9 allowed Dayana to make a distinction between different types of sinusoidal graphs. Although Dayana performed a K2f move in move 45.1, the elongated nature of her comment, combined with the pause after her move, may indicate that Dayana was not entirely in agreement with Hannah. After Dayana made the distinction between two different types of graphs, she seemed to be satisfied with Hannah’s K1 moves in exchange 8, suggesting that a cosine graph would be the appropriate graph to model the situation.

¹⁸ Nested exchanges occur when individuals engage in multiple exchanges at the same time, determined by negotiating two different objects of negotiation (Ventola, 1987).

With exchange 10, Hannah initiated a new exchange with a K1 move to establish that the graph would need to be a negative cosine graph (move 48.1). Hannah rephrased this information with move 48.2, when she specified that the graph would need to start at its minimum and then go to its maximum. Finally, Hannah added some more information to her K1 move complex by making a contrast with the type of graph that they did *not* have (move 48.3). Dayana performed a follow up move with move 49.1, in agreement with Hannah's move complex.

All of the exchanges in Table 4.3 were knowledge exchanges, indicating that the object of Negotiation in each case was some piece of information. In exchange 7, the object of Negotiation was whether or not the cosine function was the appropriate function, and then in exchanges 8 and 9 the objects of Negotiation changed to determining the different shapes of different sinusoidal graphs. During exchanges 8 and 9, Hannah and Dayana traded the K1 and K2 positions. In exchange 10, the object of Negotiation switched back to the appropriateness of the cosine function for representing the situation. This time Hannah was acting in the K1 position, as opposed to her K2 position in exchange 7.

Coding for action exchanges was useful for identifying moments when students coordinated their actions at the computer for solving the problem. In Table 4.5 I give an example of a sequence of action and knowledge exchanges between Carson and Abbey. Carson and Abbey worked on the version of the problem in which the diameter of the Ferris wheel was 140 meters and it took 40 minutes to complete one revolution. In the example in Table 4.5, Carson and Abbey were working to plot points to represent their height off the ground at 0 minutes, 10 minutes, 20 minutes, 30 minutes, and 40 minutes. They used a combination of knowledge exchanges and action exchanges to work on the problem together.

Table 4.5
Coding Action Exchanges Between Carson and Abbey

Exchange #	Turn #	Speaker	Move	Code
13	60	Abbey	1. Mm, we're gonna estimate that that's about 75. 2. And	K1 n/m
14			3. [Abbey clicks on the point several times, and then drags the point to (10,75).] 4. [Sighing] Oh my gosh.	A1 exclamation
15			5. Okay, and so then when you're 20, you're a hundred and 45. 5. And then -	K1 interrupted
16	61	Carson	1. [Carson reaches for the mouse.]	A1
	62	Abbey	1. Yes you can do that part.	A2f
17			2. And then a hundred and 45.	A2
	63	Carson	1. [Carson places a point at (20,145).]	A1
18			2. At 30? 3. [Pause 5 seconds. Carson waits for Abbey to give him an answer.]	K2 ro
	64	Abbey	1. You're back to, um, whatever the first one is, 75.	K1
	65	Carson	1. On 30?	cfrq
	66	Abbey	1. Mm hmm.	cf
19			2. So move up to where, lined up with that one. [Indicating to the point at (10,75).]	A2
	67	Carson	1. This one?	cfrq
	68	Abbey	1. Mm hmm. 2. That one, yeah.	cf cf = (68.1)
	69	Carson	1. [Carson places a point at (30,75).]	A1
20	70	Abbey	1. And then slide that one up with that one. 2. [Pointing that the point at 40 minutes should be the same height as the point at 0 minutes.]	A2 A2 + (70.1)
	71	Carson	1. [Carson moves a blue point to (40,5).]	A1
	72	Abbey	1. And that's fine for now.	A2f

The example in Table 4.5 began with Abbey performing a K1 move to make a comment about approximating where $y=75$ would be on the coordinate plane provided in the Etoys notebook. After that, Abbey performed an action move to drag and drop one of the points to plot a point at (10,75) (move 60.3). At that moment Abbey struggled, not with the mathematical content but with the act of dragging the point with the mouse and placing it appropriately on the

screen. Her exclamation in move 60.4 seemed to be an exclamation of frustration. In exchange 15 Abbey used a K1 move, offering some information, to establish where the next point would go. It seems that Abbey would have followed with an action move, but in exchange 16 Carson performed an action move to reach for the mouse (move 61.1). After Abbey conceded the mouse to Carson (move 62.1) she instructed him on where to place the next point (move 62.2).

Following his action move to place a point at (20,145), Carson initiated a knowledge exchange with move 63.2 in exchange 18. Carson's question, "at 30?" was ambiguous. He could have been asking, *what would be the height off the ground 30 minutes into the ride?* Alternatively Carson could have been asking, *where should I place the point corresponding to 30 minutes?* The nature of those two questions would be slightly different. In the first version of the question, Carson would have been explicitly requesting some information. In the second version of the question, Carson would have been requesting more explicitly some request for instruction on how he should act. I coded Carson's move as a K2 move primarily because there is no explicit move in the system of Negotiation to identify a request for a command to act. While an A2 move indicates a demand for action, and a dA1 move indicates an offer for action, there is no move to explicitly request directions on how to act. Furthermore, if he were asking what to do, Carson would still implicitly be asking the question, *what would be the height off the ground 30 minutes into the ride?* Therefore I coded his move as a K2 move, as he was requesting some information about what y-coordinate would correspond to an x-coordinate of 30. In response to Carson's K2 move, Abbey performed a K1 move to tell him that the point should be at y=75. Carson performed a move, "cfrq," to request confirmation that the point corresponding to x=30 would be at y=75, and Abbey confirmed.

Exchanges 19 and 20 were both action exchanges in which Abbey performed A2 moves to tell Carson where to place the points, and Carson performed A1 moves in response. With her A2 moves, Abbey indicated that the points corresponding to $x=30$ and $x=40$ should line up vertically with the points corresponding to $x=10$ and, respectively, $x=0$. After those two exchanges Abbey indicated, in move 72.1, that the points Carson had plotted were satisfactory, and they could move on to the next phase of the problem.

Comments on Negotiation Coding and Reliability

The examples in Tables 4.4 and 4.5 illustrate the process of coding students' transcripts according to the system of Negotiation. Each Negotiation exchange can be identified according to (a) whether it was an exchange of information or action, (b) what was the object of negotiation (the information or action to be exchanged), and (c) who was positioned as a primary or secondary knower or actor. Establishing those components in students' talk allowed for a connection between how students were positioned at any given moment and the mathematical content of their talk. Students who performed K1 moves in any given exchange were positioned as the experts within the pair for that exchange. When students performed K2 moves they positioned themselves as novices. Coding each Negotiation exchange allowed for an examination of how students' positions changed on a moment-by-moment basis.

The distinctions between experts and novices were less clear during action exchanges than during knowledge exchanges. In previous work (DeJarnette & González, 2013), we began with an assumption that students performing A2 moves would be facilitating group work in some way. The example of Carson and Abbey in Table 4.5 illustrates that students' use of A2 and A1 moves were more varied. Carson's A1 move to take the mouse allowed him to assume a position of authority within his pair. However Abbey's A2 moves to direct Carson's actions seemed to

indicate that she was positioned as the expert regarding the mathematics at hand, even though she let Carson control the mouse. Accordingly, I did not make any a priori assumptions about how students were positioned, regarding the mathematical authority within the group, based on their performance of action moves.

For my analysis, I coded interactions between students, but I did not code student-teacher or student-researcher interactions. I made this choice because I was specifically interested in answering questions regarding how students positioned themselves towards their peers, and not how students positioned themselves or were positioned within different settings within the mathematics classrooms. My examples throughout come from interactions between students, and the enumeration of exchanges refers to the exchanges between students. At moments when a hint or comment from the teacher was directly relevant to students' conversation, I will indicate that explicitly.

To establish the reliability of my coding scheme, I enlisted the work of another graduate student with whom I compared coding on a randomly selected 5-minute segment of transcript. I provided my second coder with the section of transcript, and we read through the transcript together so that the second coder could clarify what the students were talking about during the 5-minute segment. After reading the transcript together, we checked our coding for reliability in two phases. First, I had the second coder parse the transcript into moves according to the independence or dependence of clauses, and the natural flow of the conversation. The second coder matched my coding on 92% of the moves that I had parsed during the 5-minute segment. I deemed that satisfactory reliability and moved on to the second phase.

In the second phase of coding, I had the second coder code the parsed transcripts according to the moves in the system of Negotiation. For the coding, I provided the second

coder with the list of moves, and their descriptions (see Appendix G). I also provided the second coder with a summary of coding conventions (also in Appendix G). Examples of the coding conventions include notes that I used for determining a K1 versus K2 move to initiate an exchange, identifying when a new exchange has begun, and the different ways for coding responses to challenge moves. The second coder read through the conventions and asked clarification questions. To practice the coding conventions, the second coder and I coded a different 5-minute segment of transcript, which had already been parsed, together. The practice session allowed me to clarify some of the important points in coding and to make sure the second coder understood the coding conventions. After our practice session, the second coder coded the original 5-minute segment independently.

When I checked our reliability, I only counted our reliability coding synoptic moves and challenge moves. The purpose for this choice was that students' synoptic moves and challenge moves were the most critical for my analysis of how students enacted and changed positions during their work. Moves such as "request for confirmation" and "clarification" indicated instances when speakers either misheard or misunderstood each other. Although those moves were important for the smooth functioning of students' work in pairs, they were not critical to my analysis of students' positioning practices. I found that the second coder matched my coding on 87% of the synoptic and challenge moves that I had coded. Based on earlier work (Mesa & Chang, 2010), I considered that satisfactory reliability to move on with the analysis.

Results

I divided my results into three sub-sections, according to each of my three research questions. First, I examine the patterns of positioning across pairs of students in the study. Next, I provide an overview of how students enacted, and also changed, their positioning towards their

partners. Finally, I give a closer look at the interactions between students' positioning practices and the nature of their problem solving in pairs.

Patterns of Positioning Across Pairs of Students

The first step of my analysis was to answer my first research question: *What positions did students take up through their work with their peers during pair work at the computer?* Table 4.6 provides an overview of how students in the study positioned themselves through the performance of K1, K2, and A2 moves. The percentages in Table 4.6 should be read as a percent of exchanges within the pair or group. For example, Carson performed K1 moves in 20% of his 178 exchanges with Abbey over the course of the 2-day lesson. The total number of exchanges for each pair refers to the number of exchanges, related to the Etoys lesson, over the course of the two days. The percentages in each row do not necessarily add up to 100%, because students did not have to perform K1, K2, or A2 moves in every exchange. In addition, the percentages in each pair add up to more than 100%, because multiple students could each perform a move within a single exchange. For example, an exchange between Carson and Abbey could be coded as a K2 exchange for Carson and as a K1 exchange for Abbey.

Table 4.6
Summary of Students' Positioning Moves Within Their Pairs

Pair	%K1	%K2	%A2	# Exchanges
Carson Abbey	20% 40%	45% 15%	12% 37%	178
Jalisa Zach	17% 68%	21% 1%	5% 3%	101
Gia Courtney	39% 20%	23% 12%	21% 12%	132
Hannah Dayana	35% 27%	18% 16%	19% 14%	265
Shane Maya	29% 25%	7% 5%	10% 30%	124
(Bailey) Aubrey	23% 30%	16% 10%	12% 15%	83
Cara Maggie	24% 26%	10% 35%	11% 33%	250
Mike Jessa	25% 41%	8% 10%	15% 9%	145
Mitchell Reese	39% 20%	7% 19%	6% 15%	227
Tori Sean (Aubrey)	28% 30% 3%	35% 21% 9%	18% 17% 0%	301
Lucas Elizabeth Andy	16% 23% 9%	19% 11% 18%	20% 36% 21%	256

Note: Percentages should be read as a percent of exchanges within a pair. For example, Carson performed a K1 move in 20% of the 178 Negotiation exchanges between Carson and Abbey.

Students' performance of K2 moves. Students' performances of K2 moves displayed a pattern related to students' interactions between students during pair work. First, in groups where students performed higher percentages of K2 moves, students also performed more

exchanges overall. For example, Carson and Abbey, Hannah and Dayana, Bailey and Aubrey, Cara and Maggie, Mitchell and Reese, Sean and Tori, and Lucas, Elizabeth and Andy, all had at least one student who performed K2 moves in at least 15% of exchanges. In many of these pairs both students performed K2 moves in 15% or more of exchanges. In addition, these pairs of students had more exchanges overall than the other pairs of students—Jalisa and Zach, Gia and Courtney, Shane and Maya, and Mike and Jessa. The only exception to this was the pair of Bailey and Aubrey, but Bailey and Aubrey only worked together on day 1 of the lesson. One could expect that Bailey and Aubrey would have had approximately twice as many exchanges had they worked together over the course of two days. Students used K2 moves for asking questions and for suggesting ideas to be confirmed by their partners. It seems that, when students performed such moves more often, there were more discussions overall between students.

The relationship between K1 and A2 moves. The relationship between students' performance of K1 moves and A2 moves revealed three different patterns about how students used action moves to assume positions during group work. First, there were pairs in which there was an expert in the pair who also performed more A2 moves than the other member of the pair. Second, there were pairs in which there was an expert who performed the majority of K1 moves, but who did not performed as many A2 moves. Third, there were pairs in which there was no expert according to the performance of K1 moves, and students distributed A2 moves evenly among themselves. I discuss each of these in turn.

Expert performing K1 and A2 moves. Abbey and Gia were both positioned as experts in their pairs according to their performance of K1 moves. Abbey and Gia, who were not always in control of the computer, used action moves to direct their partner's work at moments when their

partners had control of the mouse or keyboard. This use of action moves was illustrated by the exchange between Abbey and Carson in Table 4.5. Abbey used three different A2 moves to instruct Carson on where to plot points to represent the height off the ground as a function of time. Abbey told Carson, “And then a hundred and 45,” which served as an instruction to place a point at (20,145). She also told Carson, “move up to where, lined up with that one,” and “then slide that one up with that one.” Abbey’s use of A2 moves in these examples were consistent with her positioning as the expert in the pair. With this use of an A2 move, the expert in the pair was able to assume some control over the work of the pair, and therefore remain the expert, even when she was not in control of the computer.

Expert performing K1 moves but not A2 moves. In the case of Zach and Mitchell, both students were positioned as the expert in the pair in terms of their performance of K1 moves, but neither student performed many A2 moves. The critical difference between the cases of Zach and Mitchell and the cases of Abbey and Gia was that Zach and Mitchell were, almost exclusively, in control of the mouse and keyboard for the duration of their work on the problem. Zach and Jalisa solved the Ferris wheel problem first on paper and then worked on inputting their solution into Etoys. Zach maintained control of the pencil and paper while working out a solution, and Zach was in control of the computer when they transitioned to working with Etoys. Mitchell and Reese solved the problem using Etoys, but Mitchell was the student who was physically in control of the computer for the entire 2-day lesson. Zach and Mitchell had little incentive to perform A2 moves, because they were performing all of the actions in their respective pairs. Students who were positioned as experts, and also in control of the computer, performed relatively few action moves towards their partners.

Evenly distributing K1 and A2 moves. When students' performances of K1 moves were more evenly distributed among the members of the pair or group, students' performances of A2 moves were also more evenly distributed. This was the case for Hannah and Dayana, Shane and Maya, Bailey and Aubrey, Mike and Jessa, Tori and Sean, and Lucas, Elizabeth, and Andy. The pair of Tori and Sean, and Aubrey on day 2 of the lesson, provides an example of the distribution of students' A2 moves between the members of different groups. On day 1 of the lesson, Tori was in control of the mouse and keyboard during most of the day. Table 4.7 illustrates a sequence of exchanges between Tori and Sean where they determined what function they would use to represent their height off the ground as a function of time.

Table 4.7
A Sequence of Exchanges Between Tori and Sean on Day 1

Exchange #	Turn #	Speaker	Move	Code
11	34	Sean	1. So it looks like we're gonna have a [pause 1 second]	n/m
	35	Tori	1. A sine graph.	K1
	36	Sean	1. A sine graph?	cfrq
	37	Tori	1. Or a cosine graph.	self-corr
12	38	Sean	1. A negative cosine graph.	K1

Both Tori (moves 35.1 and 37.1) and Sean (move 38.1) performed K1 moves to contribute ideas about what type of function they would use to model the scenario. Tori performed the initial K1 move to suggest a sinusoidal function, and she corrected herself to suggest a cosine function. Sean initiated a new exchange with a K1 move to suggest that the pair of students would use a negative cosine function to represent the scenario. The interaction in Table 4.7 is illustrative of how Tori and Sean shared the mathematical expertise through their performances of K1 moves. Neither student was always positioned as the expert. Their action

moves also illustrated a distribution of power within the group. Table 4.8 provides examples of action moves from Tori and Sean, as well as Aubrey on the second day of the lesson.

Table 4.8
A2 moves from Sean, Tori, and Aubrey

Exchange #	Turn #	Speaker	Move	Code
15			<i>[Tori has entered a cosine function into the Etoys script to represent height as a function of time.]</i>	
	74	Sean	1. All right, hit play and see what it does.	A2
251			<i>[Sean is at the computer. The group is trying to find a coefficient of the independent variable that will create a graph with the appropriate period.]</i>	
	500	Aubrey	1. Try putting 40 instead of 35,	A2
			2. cuz the period is 40,	A2 + (500.1)
			3. cuz that's where the wheel stops.	A2 + (500.1)
	501	Tori	1. Oh yeah that would make sense.	A2 = (500.1)
269			2. So try, like, 70 and 35 again.	A2

Note: Non-consecutive exchanges are separated by a bold line.

In exchange 15 from Table 4.8, Sean performed an A2 move towards Tori to direct her to test the script that she had created using the cosine function. Although Sean had previously suggested a negative cosine function after Tori's suggestion of a cosine function, Tori had entered a cosine function in the script to represent the height. By performing an A2 move, Sean contributed to the efforts of the pair even though he was not in control of the computer or positioned as the expert. Sean's A2 move provoked Tori and Sean to make a connection between the syntax and the graphical output, which then led them to revise their solution.

Exchanges 251 and 268 occurred on day 2 of the problem, when Aubrey was working with Tori and Sean. On day 2, Sean was in control of the computer for much of the time. In exchange 251, Sean had entered a coefficient of 35 inside the sine function, and Aubrey performed an A2 move complex to direct Sean to change the coefficient from 35 to 40 (moves

500.1-500.3). With her A2 move complex, Aubrey positioned herself in the conversation, and also she made an explicit connection between the construction of the function and the context of the Ferris wheel problem. Tori performed a move in agreement with Aubrey's A2 move, and Sean then responded by changing the coefficient accordingly. In exchange 268, Tori made a similar A2 move to Aubrey's, by giving Sean an instruction to revise the coefficients in their construction of the function (590.2).

When students distributed the position of expert between them according to their performance of K1 moves, they could also use A2 moves to make contributions to the work at the computer. Even though Aubrey joined Tori and Sean on the second day of the lesson, and played a relatively minor role in the conversation, Aubrey used A2 moves to engage meaningfully in the conversation. When there was no clearly defined expert in a pair or group, students also had more opportunities to perform A2 moves to instruct their peers on how to work on different parts of the problem.

There was one exception to the even distribution of K1 and A2 moves, and that came from the pair of Maggie and Cara. Maggie and Cara performed K1 moves almost equally as often (24% of exchanges versus 26%). However, Maggie performed A2 moves three times as often as Cara did (33% of exchanges versus 11%). This phenomenon between Maggie and Cara seems to have resulted from a combination of two factors. First, Cara was the student seated at the computer for most of the work on the problem. Second, Cara seemed to initiate off-task conversations, for example about an upcoming after-school activity or recalling a conversation from outside of class. Maggie engaged in those conversations at times, but she also attempted to keep the pair on-task.

Maggie made multiple comments such as, “we need to actually do this now,” and “back on topic,” to direct Cara’s attention from an off-task conversation back to their work the problem. Maggie also used comments directly related to the mathematical concepts, for example, “we just need to figure out where the amplitude goes.” Maggie’s A2 moves, in general, made points about things that Maggie and Cara needed to do as a pair. This was a contrast to, for example, the case of Abbey and Carson, when Abbey gave Carson explicit directions. Instead, Maggie used A2 moves to talk about the things that she and Cara needed to do together. Although Maggie’s comments were directed at Cara to do something or pay attention to some mathematical point, Maggie’s use of “we” in her commands maintained a sense of co-ownership over the responsibility for working on the problem. Personal pronouns have been identified as markers of positioning in mathematics classrooms (e.g., Fairclough, 2001; Herbel-Eisenmann, Wagner, & Cortes, 2010; Rowland, 2000). The pronoun “we” is something that teachers often use in mathematics classrooms, although it is often ambiguous whether “we” refers to the teacher as part of the mathematical community, the teacher and students, or some other variation (Pimm, 1987). With Maggie’s use of “we” towards Cara, she seemed to be acting in a way similar to a teacher, suggesting that both students should stay on task, without targeting Cara in particular.

Students’ Moves to Enact and Change Positions

After identifying what positions students enacted, I also examined how students enacted and changed positions towards their partners with my second research question: *How did students enact and change positions through their talk?* I aggregated the different ways that students initiated exchanges according to the four basic speech functions, requesting or demanding information or action. For this purpose, I aggregated all of the Negotiation

exchanges across all of the pairs and groups of students. Overall, students initiated the most exchanges by performing K1 moves, or in other words providing some information. Initiating exchanges with K1 moves was followed in frequency by initiating exchanges with K2 moves, then with A2 and A1 moves. To illustrate how students used these moves I give an example of each.

Initiating an exchange with a K1. Students initiated 46% of Negotiation exchanges by performing a K1 move to make a statement of some information. Table 4.9 illustrates this practice, with an example from Shane and Maya's conversation on the first day of the lesson. At the moment of the example in Table 4.9, Shane and Maya were looking at the Etoys notebook, trying to determine how they would plot the points. Shane initiated an exchange by giving some information.

Table 4.9
An Example of Shane Initiating an Exchange With a K1 Move

Exchange #	Turn #	Speaker	Move	Code
15	41	Shane	1. Oh! 2. So it's supposed to be how high off the ground.	exclamation K1
	42	Maya	1. (So we can -)	interrupted
16	43	Shane	1. (So this is the ground) so it should go like this [gesturing increasing curve] or something.	K1

In Table 4.9, Shane initiated an exchange in which the object of Negotiation was the meaning of the *y*-axis (move 41.1). Shane's statement did not come in response to any question from Maya. Prior to his statement, Shane and Maya had been sitting in silence for several seconds. They were both looking at the computer screen, and Shane spoke up without prompting. By saying, "it's supposed to be how high off the ground," Shane provided some information about the meaning of the *y*-axis in the Etoys notebook. Maya began to respond

(move 42.1), but Shane initiated a new exchange at the same time regarding the shape of the curve they should make (move 43.1). Since the object of Negotiation changed between Shane's move 41.2 and 43.1, I identified Shane's move 43.1 as a distinct exchange. Therefore, the example shows a sequence of two exchanges in which Shane initiated the exchange by positioning himself as the primary knower. Overall, students' performance of K1 moves to initiate exchanges enabled students to assume the position of expert within an exchange, without necessarily being designated as the expert by their partner.

Initiating an exchange with a K2. Students initiated 24% of all Negotiation exchanges by performing a K2 move to either ask a question or to suggest an idea to be confirmed by a peer. I provide an example of this in Table 4.10, from a conversation between Mike and Jessa. Mike and Jessa worked on the version of the problem in which the Ferris wheel took 40 minutes to make one complete revolution. At the moment of the example in Table 4.9, Mike and Jessa were working on plotting points to represent their height off the ground.

Table 4.10

An Example of Mike Initiating an Exchange With a K2 Move.

Exchange #	Turn #	Speaker	Move	Code
6	18	Mike	1. It's a 20 minute Ferris wheel?	K2
	19	Jessa	1. No,	K1
			2. cuz it takes 40 minutes to get all the way around.	K1 + (19.1)
	20	Mike	1. Forty minute Ferris wheel?	clfy
	21	Jessa	1. Yeah.	rclfy
	22	Mike	1. Oh my go:osh.	exclamation

With his K2 move in move 18.1, Mike asked a question about the context of the problem, to make a connection with how Jessa was plotting the points. By performing a K2 move, Mike positioned Jessa to answer his question by performing a K1 move. Jessa actually answered Mike's question with a K1 move complex, in which she disagreed with suggestion that the Ferris

wheel took 20 minutes (move 19.1) and also specified that it took 40 minutes to go all the way around (move 19.2). After Mike clarified that the Ferris wheel took 40 minutes (moves 20.1-21.1), he ended the exchange. By performing K2 moves to initiate exchanges, students positioned themselves as novices within an exchange and thereby positioned their partners as experts. Following such a K2 move, students' partners could determine whether or not they would take up the K1 position.

Initiating an exchange with an A2. Of all of the Negotiation exchanges, students initiated 22% of exchanges by performing an A2 move. In Table 4.7 I presented three examples of students' use of this move from the group of Sean, Tori, and Aubrey. On the first day of work, Sean provoked Tori to check her work by initiating an exchange telling Tori to, "hit play and see what it does." Both Tori and Aubrey contributed to the mathematical discussion by initiating action exchanges with A2 moves. Aubrey suggested to Sean to change a coefficient, "try putting 40 instead of 35." Tori later suggested how they could readjust coefficients after working on a different part of the problem.

Initiating an exchange with an A1. Finally, students initiated only 8% of exchanges by performing an A1 move. I gave an example of this way of initiating an exchange in the example from the work of Carson and Abbey in Table 4.4, of which I repeat a segment here. In the example in Table 4.11, Abbey began by making a statement about how they would plot points to represent the height. Carson interrupted Abbey's follow up statement by performing an action move to assume control of the mouse to work on plotting the points.

Table 4.11
An Example of Carson Initiating an Exchange With an A1 Move

Exchange #	Turn #	Speaker	Move	Code
15		Abbey	4. Okay, and so then when you're 20, you're a hundred and 45. 5. And then -	K1 interrupted
16	61	Carson	1. [Carson reaches for the mouse.]	A1
	62	Abbey	1. Yes, you can do that part.	A2f

Carson's A1 move in move 61.1 changed the object of Negotiation from how the points would be plotted (which was the object in exchange 15) to who was in control of the computer. After the brief action exchange in exchange 16, Carson and Abbey continued talking about how the points would be plotted in the plane. However, Carson's A1 move was important for enabling him to assume some agency in his and Abbey's work on the problem. Had Carson not performed a move to take the mouse, Abbey could have plotted all of the points without discussing why or how she plotted them with Carson. By initiating an action exchange with an A1 move, Carson found a way to engage in the work on the problem.

Students' Positioning Practices and Problem Solving Processes

With my third research question, I investigated more closely the relationship between students' positioning practices and their work on the Ferris wheel problem: *What types of positioning practices supported or hindered students in their problem solving processes during pair work?* To answer this final research question I sought to identify instances of collaborative productive struggle in students' work, when students collaborated to make sense of mathematics through their work on the problem. After examining students' positioning practices in terms of what patterns of positioning students displayed, and how they enacted those positions, I considered more closely how students' patterns of positioning reflected instances of collaborative productive struggle. I will first describe the different ways that students used

challenge moves towards one another, and then I will identify how challenge moves led to collaborative productive struggle between students.

Students' performance of challenge moves. The first step in identifying instances of struggle between students was to consider how students used challenge moves towards each other during their work. Table 4.12 provides an overview of the different purposes of students' challenge moves during their work on the Ferris wheel problem. I identified two categories of challenge moves that were directly related to the mathematical aspect of students' work on the Ferris wheel problem. Those two categories corresponded to the two sub-problems comprising the Ferris wheel problem: plotting points to represent one's height off the ground, and constructing a function in Etoys to represent one's height off the ground. In addition, students performed challenge moves for a variety of purposes not directly related to the mathematical content of their work. I will discuss the mathematically related challenge moves first, and then provide an overview of the other ways that students used challenge moves towards their peers.

Table 4.12

The Content of Students' Challenge Moves During the Etoys Lesson

Purpose of Challenge	Percent of Challenge Moves
<i>Directly related to mathematical content</i>	
Plotting points to represent height	15%
Constructing the function in Etoys or on paper	25%
<i>Not directly related to mathematical content</i>	
Deflect a K1 move	29%
Unrelated to work on the problem	17%
How to use the features of Etoys	12%
Physical control of computer	2%

Plotting points to represent the height off the ground. Fifteen percent of the challenge moves that students performed were related to the activity of plotting points to represent one's height off the ground. By referring to a challenge move that served the purpose of plotting

points to represent height, I mean that 15% of the challenge move students performed directly challenged a peer's statement or action to plot points in the Cartesian plane representing their height as a function of time. An example from Mitchell and Reese, presented in Table 4.13, shows how students used this type of challenge move. In this example, Mitchell had just plotted a sequence of points to represent height off the ground at 0 minutes, 10 minutes, 20 minutes, and 30 minutes. Mitchell and Reese were working on the version of the problem in which the Ferris wheel took 30 minutes, but Reese overheard Ms. Alexander talking to another group about the 40 minute version of the problem.

Table 4.13
A Challenge Move Between Reese and Mitchell About Plotting Points

Exchange #	Turn #	Speaker	Move	Code
8	40	Reese	1. I think we did that wrong.	K1
	41	Mitchell	1. Did what wrong?	clfy
	42	Reese	1. The blue dots, 2. cuz she said at 40 minutes.	relfy relfy + (42.1)
	43	Mitchell	1. Forty minutes?	check
			2. It shouldn't even go to 40 minutes,	ch-1
			3. cuz it takes 30 minutes to go all the way up and all the way down.	ch-1 + (43.2)
	44	Reese	1. So it takes 60 minutes total.	rch-1
	45	Mitchell	1. No I mean, for it to go completely all the way around, it should take 30 minutes.	ch-2
	46	Reese	1. Oh.	rch-2

Reese initiated exchange 8 with a K1 move suggesting that he and Mitchell had plotted their points incorrectly. When he made his reasoning for this suggestion clear (moves 41.1-42.2) Mitchell challenged Reese's statement by saying, "it shouldn't even go to 40 minutes" (move 43.2). That statement by Mitchell was the first of two challenges in the exchange. With his comment, Mitchell directly challenged Reese's statement that their points should include a point corresponding to 40 minutes. When Reese responded to Mitchell's challenge by suggesting that

it may take 60 minutes total (move 44.1), Mitchell rephrased his initial challenge with a new challenge move (45.1), saying more explicitly that it would take 30 minutes for the Ferris wheel to make one complete revolution.

Challenge moves between students regarding how they would plot the points were critical to their mathematical work. Plotting the points to represent height as a function of time was what allowed students to identify which function they would need to use, based on the shape of the graph. Plotting the points also eventually allowed students to determine the appropriate parameters of that function. Only 15% of challenge moves were devoted to the idea of plotting points, which suggests that the process of plotting points may have been more straightforward for students than other aspects of the problem. These challenge moves were important, not only in terms of students' positioning, but also in terms of creating opportunities for students to make their mathematical ideas more explicit in their conversations with their partners.

Constructing the function in Etoys. Of all of the challenge moves that students performed, 25% of those challenge moves were directed at the mathematical concepts related to constructing the sine or cosine function that would represent the situation. Constructing the sine or cosine function was the second problem in students' work, after they had plotted a set of discrete points. These challenge moves included moves about the necessary amplitude, period, and shift of the sinusoidal graph and the connections between them. Challenge moves regarding the construction of the function constituted the second greatest percentage of challenge moves.

An example of using a challenge move to challenge the construction of a function can be found in Table 4.14. In this example, Tori and Sean were working on adjusting the coefficient of their cosine function to create a graph with the appropriate amplitude. On the first day of their work on the problem, Tori and Sean had decided that they would use a coefficient of 70 to

construct a function that stretched vertically up to 140. At the moment of the example in Table 4.14, Sean and Tori were using a coefficient of 35, which was creating a graph with a maximum value of 70. Tori initiated the exchange with a K1 move stating what the maximum value of the graph should be, based on their work from the previous day.

Table 4.14
A Challenge Move Between Sean and Tori About Constructing Their Function

Exchange #	Turn #	Speaker	Move	Code
14	83	Tori	2. the maximum is 140, right?	K1
	84	Sean	1. Mm hmm.	bch
			2. If the maximum's 140 then that's wrong.	ch
	85	Tori	1. No.	rch
	86	Sean	1. That'd be a maximum of 70.	corr
	87	Tori	1. True.	rcorr

In response to Tori's K1 move saying that, "the maximum is 140," (move 83.2), Sean performed a challenge move (84.2) to argue that if the maximum value of the graph was 140, then the function they had just constructed was incorrect. Actually, Tori was correct in her statement that the maximum value of the graph should have been 140. The function Tori and Sean were working with at this moment was not correct. However, by challenging Tori, Sean repositioned himself as the person with the mathematical authority within the exchange. Following his challenge move, Sean performed a move to correct Tori, by telling her, "that'd be a maximum of 70" (move 86.1). Tori responded to Sean's correction by agreeing with him and ending the exchange. Similarly to challenge moves regarding how to plot points, students' challenge moves about how they would construct their sinusoidal functions marked opportunities for students to make their thinking explicit, reason with one another, and reach some consensus about the mathematics at hand.

Challenges moves regarding how to use Etoys. Challenge moves regarding how to use Etoys made up only 12% of challenge moves overall. These challenge moves were not directly related to the mathematical aspects of students' work, but instead they were challenges regarding how to use Etoys. Because none of the students in the study has experience using Etoys, it is not surprising that students rarely challenged one another about how to use Etoys. One could expect that students rarely felt as though they had enough expertise regarding how to use Etoys to challenge something that a peer was doing. The ways that students challenged one another regarding Etoys was likely important for the way they appropriated the tools of Etoys. For this reason, an examination of such challenge moves would be especially relevant for a study of how students' interactions may have supported their instrumented activity.

Challenge moves to deflect a K1 move. The most frequent way that students used challenge moves was to deflect a K1 move, or in other words to reject the K1 position within an exchange. Students deflected K1 moves with 29% of their challenge moves. This type of challenge move occurred when one student in a pair performed a K2 move towards another student, thereby positioning the other student as the expert to answer the question. For example, on the first day of the Etoys lesson, Mitchell and Reese constructed an initial function, and they found that the function plotted a straight line. Reese performed a K2 move towards Mitchell, asking, "Why is it just going straight?" Rather than providing an answer, Mitchell responded, "I don't know." By asking the question, Reese positioned Mitchell as the expert to explain why their graph was plotting a straight line rather than a curve. But rather than assuming that position, Mitchell deflected the move with a challenge. Consistent with a prior study of students' positioning during small group work (DeJarnette & González, 2013, April), students used challenge moves in their pairs at times as a way to avoid assuming the position of expert.

Challenge moves regarding the physical control of the computer. In addition to deflecting K1 moves, students used 2% of their challenge moves to challenge the physical control of the computer. Challenging the physical control of the computer was a move that happened only a few times among all of the pairs of students across the two day lesson. For example, while Jessa and Mike were discussing what would be the appropriate coefficient to account for the period of the function on the second day of the lesson, Mike reached for the mouse. Jessa reacted with the statement, “No. Let me try this.” Rather than allowing Mike to perform the action of using the mouse, Jessa challenged his action and assumed control of the mouse. Jessa’s challenge move did not seem to be an intentional challenge to Mike’s authority, but rather an instinctive reaction based on an idea she had. Still, Jessa’s challenge achieved the purpose of shifting positions so that she had physical control of the computer. Challenge moves regarding physical control of the computer were distinct from challenges about the mathematical content or use of Etoys, because they did not target any specific idea or strategy for solving the problem or using the software. Rather, these moves only responded to an action move to take control of the mouse or keyboard.

Challenge moves unrelated to mathematics. Finally, challenge moves unrelated to the mathematics content made up 17% of the total challenge moves between students in the study. Moves unrelated to the mathematics content included comments such as, “I don’t care,” or “you don’t really know.” For example, on the first day of the lesson, Maggie and Cara began constructing a cosine function to represent their height off the ground. Maggie made a comment that they would need to use a formula for the period of 40 minutes, and Cara responded, “you don’t really know.” Cara made her comment in a tone that sounded as though she was joking, and both students laughed following the comment. Although Cara’s statement served to

challenge Maggie's position as the expert, neither student seemed to take the challenge move seriously, and they continued with their work. I identified challenge moves as unrelated to the mathematical content when they did not identify a specific concept, idea, or strategy for working on the problem. These challenge moves likely served an interpersonal function of shifting students' positions towards one another, but they were generally not specific enough to contribute meaningfully to the discussion about the problem.

Overall, students performed challenge moves to contribute to the mathematical aspects of their work, to manage their use of Etoys and control of the computer, and for interpersonal purposes not directly related to their work on the problem. All of the different ways in which students challenged one another contributed to students' positioning. Moreover, all of students' challenge moves likely had some impact, either directly or indirectly, on students' mathematical learning through their work in pairs. An examination of the effects of students' challenge moves which were not directly related to their mathematical work is beyond the scope of this paper, although it is worthy of future examination. I focus now on students' challenge moves that were specifically related to the mathematical aspects of their work, either plotting points to represent their height off the ground or constructing their functions in Etoys. The purpose of this is to establish connections between students' performance of challenge moves and instances of collaborative productive struggle over mathematical ideas.

Connections between challenge moves and productive struggle. In the definition by Hiebert and Grouws (2007), struggle is not a process that requires multiple interactants. For the purposes of this study, I sought evidence specifically of how students collaborated to make sense of mathematics. Based on my observations of students' positioning moves, I developed an

operational definition of what I called *collaborative productive struggle*. An instance of collaborative productive struggle, or CPS, occurs when

1. A student performs a challenge move regarding some mathematical concept or idea;
2. In a sequence of exchanges referring back to the concept or idea at play in the challenge, both students contribute K2 or K1 moves;
3. The mathematical concept or idea at play is eventually correctly used, from the perspective of an expert observer.

This definition of CPS emerged from my examinations of students' work together in pairs, and it accounts for three critical features. First, an instance of CPS must be the result of *struggle about a mathematical idea* between two students. Given that I was using the system of Negotiation, I used challenge moves between students as a proxy for identifying instances of struggle.

Challenge moves about mathematical ideas were made up of the two categories of moves that I discussed in the previous section, challenge moves regarding plotting points and regarding constructing an appropriate function. By performing a challenge move about a mathematical idea, students created opportunity to resolve that challenge and thereby develop a shared sense of ownership over the mathematics at hand.

Second, an instance of collaborative productive struggle requires that, after the challenge move, the students perform a sequence of exchanges where they *refer back to the content of the challenge move*. Students refer back to challenge moves through elements of *cohesion*, which is part of the textual metafunction of speech (Halliday & Matthiessen, 2004; Halliday & Hasan, 1976). Most relevant for this study were students' use of *substitutions*, when they replaced the original object of Negotiation with a pronoun. In addition, students used *conjunctions* to connect

ideas and achieve cohesion¹⁹. Within this study, the practice of establishing cohesion was more important than the specific linguistic resources that students used to do so. Students used elements of cohesion to refer back to the content of a challenge move in proceeding exchanges. In addition to referring back to the content of the challenge, to achieve collaborative productive struggle students must both perform either K2 or K1 moves. This requirement ensures that students' struggle is collaborative, or that students establish some shared ownership through the sequence of exchanges.

Finally, in an instance of CPS, the mathematical concept or idea at play must be *eventually used correctly*, which guarantees that the struggle is productive. Whether the concept is used correctly is judged by an expert observer. I would judge an idea as used correctly if it were used consistently with standard mathematical definitions and practice. It is possible that students could resolve a challenge in a way that they think is correct, even if it is not correct from an observer's perspective. In such a case, I would not identify the struggle as productive. On the other hand, students may go through many iterations of incorrect resolutions to a challenge before eventually reaching a correct idea, as is illustrated in an upcoming example. As long as students eventually reach a correct resolution I would identify the struggle as productive, with the perspective that discussions about incorrect ideas can contribute to mathematical learning.

Instances of collaborative productive struggle can be best illustrated by examples. The first example comes from a conversation between Hannah and Dayana, which I first presented in Table 4.3, and which I include here as Table 4.15. In the example, Hannah and Dayana were debating whether they would use a sine or a cosine graph to represent their height off the ground

¹⁹ Cohesion can be achieved through five different elements of speech: conjunctions, references, substitutions, ellipses, and lexical cohesion (Halliday & Hasan, 1976). Elements of substitution and conjunction were by far the most prevalent in students' talk.

as a function of time while riding the Ferris wheel. After Hannah suggested a cosine graph, Dayana performed a challenge move to suggest that the appropriate graph would be a sine graph. In Table 4.15, I highlight the content of the challenge move, which was related to constructing the appropriate function to solve the problem. I also mark, in the following moves, where Hannah and Dayana used substitution or conjunction to refer back to the content of the challenge.

Table 4.15
Tracing the Content of a Challenge Move in the Work of Hannah and Dayana

Exchange #	Turn #	Speaker	Move	Code	Type of cohesion
8		Hannah	4. And then you start here [pointing approximately to (0,5) on the plane], 5. and then you go up, like over time, it goes up, 6. (and then it goes back down.)	K1 K1 + (42.2) K1 + (42.2)	Choice of sine versus cosine graph substitution
	43	Dayana	1. (Wouldn't that be sine?)	ch	
	44	Hannah	1. No because it's starting at the bottom and not at the middle.	rch	
	45	Dayana	1. Oh:h. [Pause 3 seconds.]	K2f n/m	
9			2. It goes up then down (then up.)	K1	substitution
	46	Hannah	1. (Mm hmm.)	K2f	
8	47	Dayana	1. Okay.	K2f = (45.1)	
10	48	Hannah	1. So it 'd be cosine, but with a negative amplitude, 2. so it would start at the bottom and then go up. 3. Instead of [gesturing with hands a curve that starts at the top then goes down and back up].	K1 K1 = (48.1) K1 + (48.2)	substitution substitution conjunction
	49	Dayana	1. Okay.	K2f	

Note: The asterisk in the rightmost column indicates moves where students referred back to the content of the challenge move.

After Dayana performed a challenge move in exchange 8, Hannah and Dayana continued to debate the choice of a sine versus a cosine function in exchanges 8, 9, and 10. The sequence of exchanges satisfies the definition of a collaborative productive struggle between the two students. First, Dayana's challenge move targeted specific mathematical content related to solving the problem. Second, both students positioned themselves in the expert position by performing K1 moves in the exchanges following the challenge move. Hannah performed K1 moves in exchanges 8 and 10, and Dayana performed the K1 move in exchange 9. Third, the students resolved the challenge in a way that was correct for solving the problem when they decided that they would use a cosine graph. Actually, Hannah and Dayana could have decided on either a sine or a cosine graph and their resolution would have been correct in the context of the problem (because a sine graph is a horizontal shift of a cosine graph). The example from Hannah and Dayana is an example of collaborative productive struggle when the students resolved the content of the challenge correctly in the exchanges immediately following the challenge.

The example from Table 4.12, when Reese made a statement that he and Mitchell had done something incorrect, and Mitchell challenged him, was also an instance of collaborative productive struggle. I repeat the example from Table 4.12 in Table 4.16, and I include the exchange that immediately followed the exchange containing the challenge move.

Table 4.16
An Instance of Productive Struggle Between Reese and Mitchell

Exchange #	Turn #	Speaker	Move	Code	Content
8	40	Reese	1. I think we did that wrong.	K1	
	41	Mitchell	1. Did what wrong?	clfy	
	42	Reese	1. The blue dots,	rlfy	
			2. 'cuz she said at 40 minutes.	rlfy + (42.1)	
	43	Mitchell	1. Forty minutes?	check	The domain of values for plotting points
			2. It shouldn't even go to 40 minutes,	ch-1	substitution
			3. 'cuz it takes 30 minutes to go all the way up and all the way down.	ch-1 + (43.2)	substitution
	44	Reese	1. So it takes 60 minutes total.	rch-1	substitution
	45	Mitchell	1. No I mean, for it to go completely all the way around, it should take 30 minutes.	ch-2	substitution
	46	Reese	1. Oh.	rch-2	
			[Pause 5 seconds.]		
9			2. It takes 30,	K1	substitution
			3. so 15 should be the highest.	K1 + (46.2)	conjunction
	47	Mitchell	1. Yeah.	K2f	

The content of Mitchell's challenge in move 43.2 was the appropriate domain of values for plotting points to represent their height off the ground. Following Mitchell's challenge, Reese responded to the challenge suggesting he still thought they needed a larger domain (move 44.1). Mitchell challenged Reese again, referring back to the same mathematical content, insisting that the entire trip on the Ferris wheel would take 30 minutes. Had Mitchell and Reese's discussion ended at the conclusion of exchange 8, I would not have identified it as an instance of collaborative productive struggle. At the end of exchange 8, a challenge move had been performed, and it seemingly had been resolved correctly. However, Reese had not

contributed to a shared understanding between the two students with either a K2 or a K1 move. Reese's comment of "oh," at the end of exchange 8 would not be enough evidence that he had assumed any ownership over the mathematical idea at hand. After a pause of several seconds, however, Reese initiated a new exchange on the same topic. Reese performed a K1 move complex stating that, since the Ferris wheel takes 30 minutes to complete a revolution, the highest point would occur 15 minutes into the ride. Reese's move complex in exchange 9 was important for revealing that he shared in the understanding of the problem. By performing K1 moves, Reese assumed ownership over the mathematical ideas that had been the content of the challenge move, and therefore I identified the interaction as an instance of collaborative productive struggle.

Not all instances of CPS included a resolution immediately following a challenge move. In the example from Table 4.14, Tori challenged Sean regarding the appropriate coefficient of the sine function to give the appropriate amplitude of the corresponding graph. Figure 4.4 illustrates the sequence of exchanges that followed from Sean's initial challenge move. The exchanges in Figure 4.4 progress from left to right, and they spanned most of the second day of the lesson. Following the challenge move regarding the coefficient of the cosine function, and its relation to the amplitude of the graph, Tori and Sean shared 14 exchanges in which they negotiated the same mathematical content from the challenge move. Those 14 exchanges did not all follow directly after the challenge move. Instead, Tori and Sean came back to the idea of the coefficient of the function and the amplitude of the graph several different times over the course of their work on day 2 of the Etoys lesson.

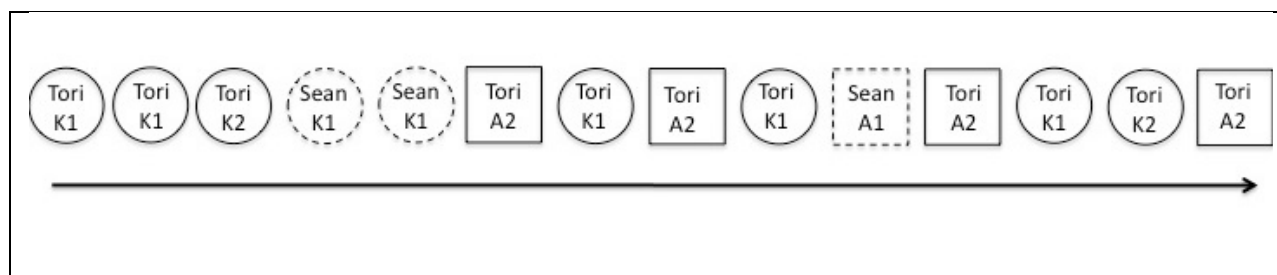


Figure 4.4. An instance of collaborative productive struggle, made up of 14 exchanges, following Tori's challenge to Sean. Squares indicate action exchanges, while circles indicate knowledge exchanges.

In Figure 4.4, both Tori and Sean performed knowledge and action moves. Tori and Sean both contributed to the sequence of exchanges with either K2 or K1 moves. The combination of the various acts of positioning between the two students ensured that their struggle was collaborative, in that both students contributed in meaningful ways to the mathematical ideas at hand. Moreover, in the final exchange of the sequence, Tori performed an A2 move to tell Sean to change the coefficient in their script to 70 (instead of 35, which it had been at the start of the sequence). When Sean followed Tori's instruction and changed the coefficient, Tori and Sean resolved the issue of the coefficient of the cosine function in a way that was correct for solving the problem. As a result of Sean's challenge move, Tori and Sean engaged in collaborative productive struggle to make connections between the coefficient in the symbolic representation of a function and the graphical output of the syntax. Their struggle persisted throughout most of the second day of the lesson, going back and forth between this and other mathematical ideas.

Discussion

The ways that students positioned themselves and their peers in this study revealed two important points about the nature of students' positioning and students' opportunities for collaborative productive struggle. The first point is that students' performance of K2 and A2 moves seemed to create opportunities for students in a pair to contribute ideas to the

mathematical discussion. The second point is that, by challenging one another, students created opportunities to engage in collaborative learning with their peers through collaborative productive struggle.

Contributing to Pair Work Through K2 and A2 Moves

While working together at the computer, students' acts of positioning served multiple, and multi-layered, purposes. Students' performance of K2 moves served multiple purposes in their work with their partners. By performing K2 moves, students were able to bring mathematical ideas to the table for discussion, and they were able to ask questions about their work on the problem. In addition, students' performance of K2 moves created opportunities to generate conversation, as was reflected by the higher number of Negotiation exchanges overall in pairs where students performed K2 moves more frequently. Finally, students who were not positioned as experts performed K2 moves to contribute to the mathematical conversation in meaningful ways. For example, in the case of Mitchell and Reese, Mitchell was most often positioned as the expert and was also in control of the computer for the majority of the lesson. By performing K2 moves, Reese created opportunities to engage in the mathematical discussion. Moreover, Reese's K2 moves were important mathematically, in that they provoked Mitchell to make his reasoning more explicit, which often led to revised solution strategies.

Research on students' positions as experts or novices has typically assumed that students who are positioned as novices participate in a less meaningful way in the mathematics at hand during group work (e.g., DeJarnette & González, 2013, April; Esmonde, 2009b). With this study, I have uncovered the more nuanced way that students used K2 moves to participate in mathematical discussions with their peers. More than just asking questions, performing K2 moves allowed students to contribute mathematical ideas and spur conversations about the

problem solving activities. This finding is reminiscent of the work of Esmonde (2009b), who found that in circumstances where students were working on novel tasks, experts and novices could contribute to a collaborative effort more equitably than in more traditional tasks. Although Esmonde's study came from a more typical group work context, with groups of 4-5 and no computers, her findings suggested that experts and novices had potential to collaborate equitably when they were removed from their typical classroom routines. In this study, students were certainly removed from their typical classroom routines, in that they were working on a problem with the use of a programming environment. In this unfamiliar setting, students' positions as experts and novices seemed to be more fluid, and students positioned as novices were able to contribute to the discussions within the pair.

Students' performance of action moves suggested certain ways in which pair work with the use of a computer may look differently than group work in more typical classroom settings. Although students initiated exchanges with A1 moves infrequently (only 8% of exchanges), students used A2 moves in a variety of ways to coordinate their work at the computer. Specifically, students could use action moves in ways that either distributed authority among peers or reinforced expert and novice distinctions between students. When students were positioned as experts in terms of their knowledge moves, they often performed A2 moves to direct the work of the novice seated at the computer. When students distributed K1 moves evenly between themselves, A2 moves served more as a way for students to contribute to the work at the computer rather than as a way to control the work at the computer. In an examination of pre-service teachers, Gerardo and Gutiérrez (2013, April) have found that patterns of conversation have potential to blur distinctions between experts and novices, creating spaces where each person has potential to contribute expertise. I see a similar phenomenon

regarding students' collaboration around the computer, where action moves become especially important for coordinating that work. Patterns in students' performance of knowledge and action moves can potentially compound inequitable interactions between experts and novices, particularly if the expert uses action moves to reinforce his or her position as the expert. However, in a more ideal scenario, action moves create more space for different students to contribute expertise at different moments during their work. Distinctions between experts and novices become less apparent if there are more ways for students to assume authority among their peers.

In light of this point, it is important to keep in mind that students in this study did not have explicit instructions for how they should divide or manage the control of the computer. Students' use of action moves depended to some degree on which student was seated at the computer, and whether students traded places at the computer over the course of their work on the problem. This study raises an interesting question about how the principles and practices of *pair programming* can be translated to mathematics classrooms in ways that support students to share mathematical authority. For example, assigning students to act as the "driver" and "observer" (Williams et al., 2002; Williams & Kessler, 2003) can support equitable interactions, but those assignments should take into account the ways students position themselves relative to the computer and relative to the mathematics. Pair programming was originally developed for industry, and it has been shown to result in shorter and more efficient programs than when individuals work alone (Cockburn & Williams, 2001; Williams & Kessler, 2003). However, it is not immediately clear that the practices of pair programming will translate to equitable interactions and collaborative learning of mathematics. When students work together around the computer to learn mathematics, it is necessary to examine how students position themselves with

regards to the physical control of the computer, their performance of action moves, and their performance of knowledge moves. Managing these different dimensions of students' interactions will contribute to finding more ways for students to collaborate during pair work at the computer.

Reflecting on all of the ways that students performed knowledge and action moves, this study emphasizes the highly dynamic nature of students' positioning when working together with peers. Other studies of positioning practices in mathematics classrooms have uncovered how restricted students' positions can be in whole classroom settings, where students are positioned as under the authority of a teacher and a textbook (Herbel-Eisenmann, 2007; Herbel-Eisenmann & Wagner, 2007, 2010). In traditional classroom settings, students have few opportunities to position themselves, but instead are positioned in certain ways by classroom practices. The ways that students use action moves in a group work setting allow students to make many different choices regarding how they will facilitate the work of their group (DeJarnette & González, 2013, April). With this study, I have found more ways in which students use K2 and A2 moves to contribute to discussions about mathematics and collaborate with their peers. Although I had expected that students' opportunities to position themselves may have been limited with only two students, I found a multitude of ways that students used acts of positioning to steer their problem solving efforts.

Challenge Moves and Collaborative Productive Struggle

The second major point about students' positioning practices is that students' challenge moves created opportunities for collaboration through instances of collaborative productive struggle between students. When students challenged one another regarding mathematical ideas for solving the problem they made choices, either explicitly or implicitly, about how they would

respond to those challenges. By examining students' performance of challenge moves, and their reactions to challenge moves, I formulated an operational definition of collaborative productive struggle. Struggle in mathematics classrooms should serve more purpose than making students feel frustrated or overly challenged (Hiebert & Grouws, 2007). Struggle should allow students to gain new mathematical insights and to make progress towards solving a problem that is within reach. Because a primary goal of this work is to create opportunities for collaboration, I argue that collaborative productive struggle should also allow two students within a pair to contribute ideas related to a mathematical concept.

Following a challenge move regarding mathematical content, there were many different paths that students could take in their exchanges with their peers. Figure 4.5 outlines a model of the multiple paths that students could take, beginning with a challenge move represented by the square figure. The different shapes in Figure 4.5 represent alternatives at the same level during students' conversations. The options enclosed in rectangular figures indicate the three options that students could follow immediately following a challenge move. First, the challenge move could be ignored or passed over, in which case the student being challenged did not respond to or acknowledge the challenge move. Ignoring a challenge move shut down opportunities for productive struggle, and therefore ended the path.

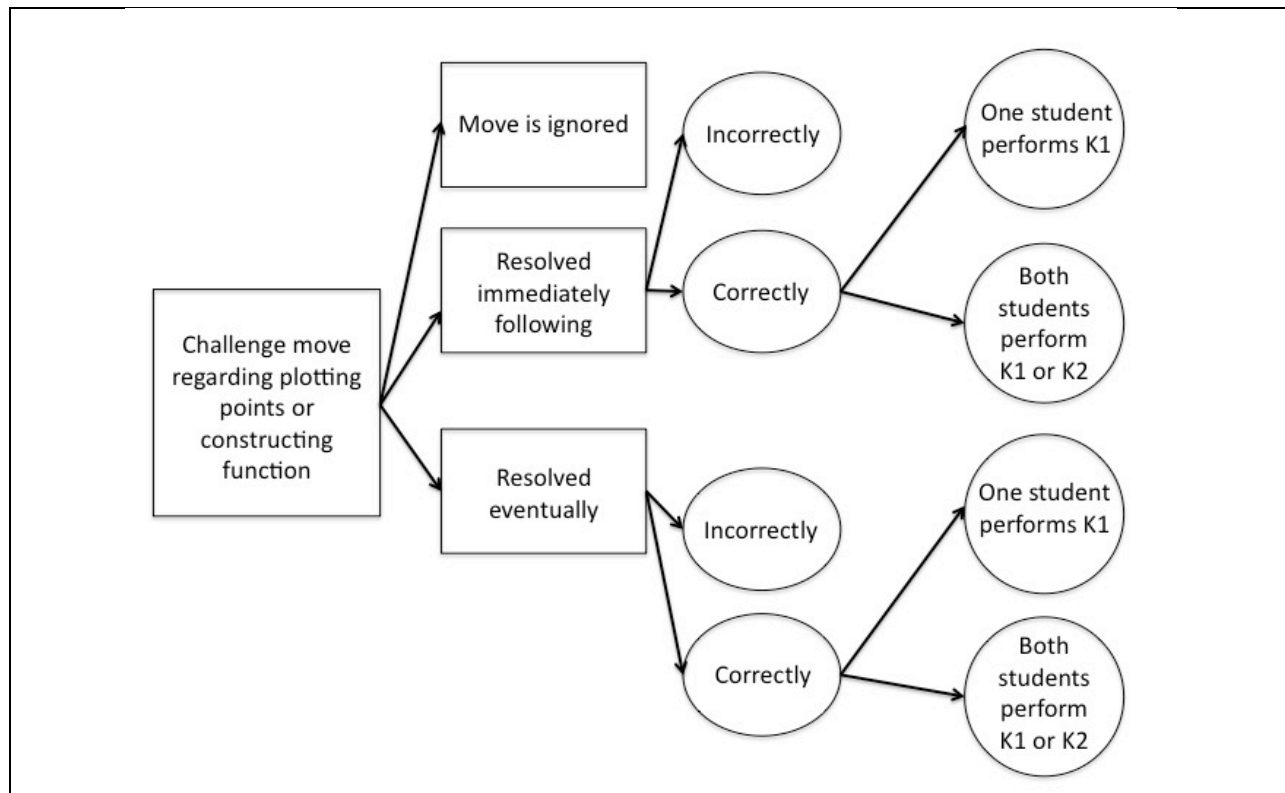


Figure 4.5. A model of conversational paths following a challenge move regarding mathematical content.

The other two alternatives for responding to a challenge move created opportunities for collaborative productive struggle between students. These two alternatives both included resolving the content of the challenge move. This resolution could come immediately following the challenge move, or it could come eventually following the challenge move. The cases of Hannah and Dayana, as well as Mitchell and Reese, gave examples of students resolving the content of the challenge move directly after the challenge occurred. In the case of Tori and Sean, the move was resolved eventually, although not directly after the move. For the struggle to be productive, students would need to resolve the mathematical content correctly, indicated in the oval shapes in Figure 4.5. If students resolved the challenge move in a way that was incorrect, and never returned to the content to correct it, then the sequences of exchanges did not constitute collaborative productive struggle.

When students correctly resolved the mathematical content of a challenge move, either immediately or eventually, the incidence of CPS still depended on whether or not they collaborated in their endeavors to resolve the content. The circles in Figure 4.5 present alternatives for how students may have positioned themselves in the exchanges where they resolved the mathematical content related to a challenge move. If one student performed all the K1 moves, and the other student performed no K1 or K2 moves, then the sequence of exchanges did not provide enough evidence of collaboration to be considered collaborative productive struggle. When both students performed either K2 or K1 moves, I considered the sequence an example of collaborative productive struggle. Figure 4.6 provides a more focused representation of the two options for students to engage in collaborative productive struggle. The only difference between those two options was whether students resolved the mathematical content immediately following the challenge, or eventually after a series of exchanges.

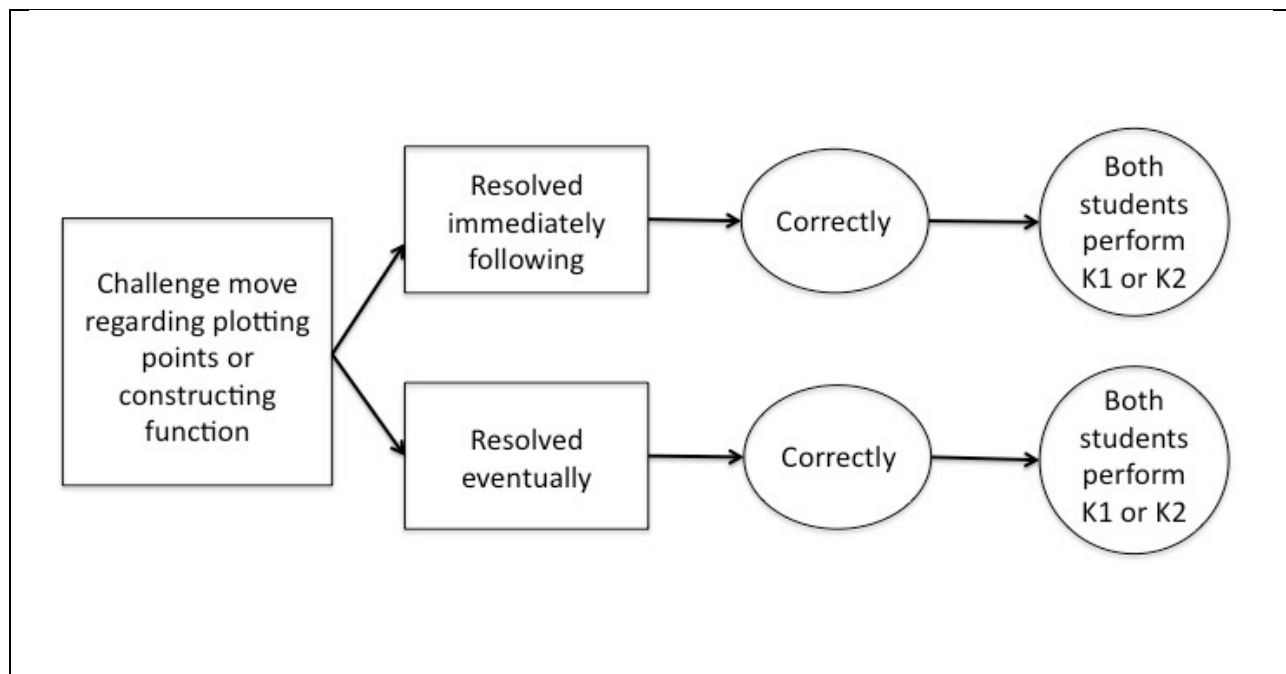


Figure 4.6. A model of collaborative productive struggle between pairs of students.

There are two important points about the model of CPS in figures 4.5 and 4.6. First, I identify an instance of CPS as beginning with a challenge move regarding the mathematical content of the conversation, but I do not make any assumption about whether the challenge is valid. In the example of Tori and Sean, Sean challenged Tori even though Tori's original statement had been correct. One could argue that, since Tori already seemingly knew the correct answer, their ongoing discussion may not have been an instance of CPS. However, by challenging Tori, Sean provoked a discussion that spanned 14 different exchanges, to which he and Tori both contributed. By the end of those exchanges, Sean came to understand Tori's point of view. Since Tori was engaged in the discussion, and her participation in the exchanges allowed her to clarify her argument and convince Sean of her correctness, the sequence of exchanges was collaborative for both students. The model of CPS does not have to begin with a challenge that is valid, or in other words a move that correctly challenges an incorrect statement. Even an incorrect challenge creates an opportunity for CPS.

Another important point about the model of CPS is that the judgment about the correctness of the eventual resolution to the challenge move is an external judgment, not a judgment from students themselves. Students could resolve a challenge move with an answer that they think is correct, but that is not correct according to standard mathematical practice. As a measure of whether students' struggle is *productive*, I relied on my own judgment of whether students' resolutions were correct. This choice has both advantages and disadvantages. The primary disadvantage of this choice is that it changes the focus of analysis from what occurs in the conversation according to the students' point of view to what occurs according to the observer's point of view. The analysis with the system of Negotiation offers an *emic* perspective, because it shows, from the point of view of the participants in the conversation,

what was the object of Negotiation and how students positioned themselves²⁰. With the analysis of the correctness or incorrectness of students' solutions, I imposed an *etic* perspective on students' work by offering an observer's analysis of students' work. This decision is not inherently problematic, but it privileges the observations of an outside observer over how students saw their own work as it related to their positions. The primary advantage of the *etic* perspective of the correctness of students' work was that it allowed me to bridge the gap between students' positioning practices and the productivity of students' work accordingly to whether they were gaining mathematical insight that would support them to establish an acceptable solution to the problem.

The ways that students engaged in collaborative productive struggle reflected a finding of Barron (2000, 2003) that groups who were successful in problem solving were able to regain joint, focused attention at solution-critical times. In Barron's work, she found that it was acceptable for students to get off track or leave a particular idea, as long as students in a group were capable of refocusing their attention on that idea at a later time. The example from Tori and Sean, who returned to the idea of the amplitude of their cosine graph with 14 different exchanges over the course of a class period, reflected the ability to return their focus to an aspect of the problem. This pattern supports Barron's notion of the importance of regaining joint attention to make progress in problem solving efforts. I would also argue that students like Tori and Sean benefitted not only from regaining attention, but also by both contributing to the conversation with K2 or K1 moves. By making equal contributions following a challenge move,

²⁰ Even though I, as an observer, performed the analysis of students' positions, I consider this as an *emic* perspective because I identified students' positions as they assigned those positions relative to each other. The positions of expert and novice, for example, referred to how students positioned themselves, not to whether or not students could objectively be considered "experts" in the mathematical content.

students gave evidence that they did in fact have joint attention to the problem, and they were both contributing to a shared mathematical understanding within the group.

With this study, I have not uncovered how factors such as gender, race, prior achievement, social or academic status within the class impact students' positioning practices or how students respond to the positions their peers take up. On one hand, by giving focus strictly to students' moment-by-moment positioning, I was able to foreground students' agency to assume positions through their talk. On the other hand, I cannot assume that all students had the opportunities to position themselves in the ways that they wanted during pair work. Settings in which students are expected to communicate and rely on the expertise of their peers are likely to privilege certain students over others (Chizhik, 2001; Lubienski, 2000; Moschkovich, 1999; Murrell, 1994; Zahner & Moschkovich, 2011; Zevenbergen, 2000). With a model for understanding how students engage in collaborative productive struggle, it is now possible to pursue research to compare how different students have or create opportunities to engage in this activity. Comparisons of different groups of students, in different classrooms and schools, can shed light on what social factors may be important predictors of students' collaboration while working together around computers.

The ways that students exchange information and actions, and the model of collaborative productive struggle, have implications for how teachers may support students' interactions during pair work. First, given that K2 moves give students a way to provoke or enter a conversation, there is an implication that asking questions creates opportunities for mathematical discussion during group and pair work. In earlier research, Webb (1989, 1991) has identified the importance of questions, specifically for students who are low-achievers in groups. Based on this study, it seems that asking questions is beneficial for more than the mathematical knowledge

students gain in response. Asking questions serves a purpose during pair work to provoke conversation, which benefits everyone in a pair or group.

Based on my observations of students' action moves, there is an implication that action moves should be used to contribute to, but not to control, work at the computer. Students are not likely to benefit from pair work around a computer when one student identifies himself or herself as the director and controls all the work (Hoyles et al., 1994). Based on work in computer science, however, students may benefit from trading roles as a driver and observer, as long as students fulfill the different responsibilities of the roles at different times (Williams et al., 2001; Williams & Kessler, 2003). There is still more work to be done to understand how role assignments can support students to share authority of a computer and mathematical ideas in ways that support collaboration. However, there is an indication from this work that action moves should not enable any one student to direct the work of another student. From the perspective that students should share positions of expert and novice, norms for collaboration around a computer should establish ways for students to contribute to work without controlling it through their performance of action moves.

In addition, the framework of collaborative productive struggle makes salient the implication that challenge moves create opportunities for collaborative learning during pair and group work. Challenging initial ideas is part of the practice of doing mathematics, and students should learn how to challenge their peers respectfully and respond to those challenges (Cohen, 1994a; Horn, 2012). Students challenge one another in many different ways, targeting either mathematical content, aspects of technology tools, or interpersonal factors. In settings where students work with technology tools, there is opportunity to understand better the impact of students' challenge moves on creating opportunities for collaborative productive struggle.

However, teachers can establish norms for students regarding when it is appropriate to challenge peers, and how they should present and respond to those challenge moves. Given the potential positive outcomes of challenge moves, teachers and students will benefit from developing practices for challenging one another during mathematical discussions.

Conclusion

In this study, I have examined students' positioning practices while working in pairs with a computer-programming environment to solve a problem in Algebra 2. The evidence has come from 10 pairs of students, and one group of three, working together over the course of a 2-day lesson in which they used a computer-programming environment to represent a sinusoidal phenomenon. The pairs of students in this study revealed the many different ways that students used positioning moves, including K1, K2, and A2 moves, to accomplish their work on the computer and to solve the problem. K2 and A2 moves were especially important to students' work in this study. Students' performance of K2 moves created opportunities for mathematical discussion. Students performed A2 moves in ways that distributed authority within a pair or group, or in ways that reinforced distinctions between experts and novices. Working together at the computer increases the complexity of students' collaborative efforts, because using the computer gives students more ways to participate in the work of doing mathematics. By giving attention to students' interactions, through their work on the problem, I have illustrated the many different ways that students position themselves and are positioned in this context.

The ways that students performed challenge moves towards their peers also had important mathematical and interpersonal implications. By focusing on the ways that students challenged one another regarding their mathematical work at the computer, I have begun to uncover how students participate in productive struggle through work with their peers. Students

did not always use challenge moves in ways that were directly relevant to their mathematical work. However, when they did, they created opportunities to engage in *collaborative productive struggle*, collaboration with their partners that led towards new mathematical insight. I see the construct of collaborative productive struggle as useful because it provides a way to identify, on a small scale, instances of a much larger process of collaborative learning. Instances of collaborative productive struggle can build upon one another, over time, to contribute to collaborative learning between students on a much larger scale.

With each of these findings, it is important to keep in mind that collaboration cannot be taken for granted as a product of assigning students to work together. Collaborative learning is a phenomenon that emerges when students increasingly participate in mathematical practices through a process of contributing to a shared product within a pair or group of peers (Staples, 2007). Collaboration should result in a sense of co-ownership over mathematical ideas and practices. However, the processes by which that co-ownership should emerge may not be apparent to students. This study contributes to scholarship that seeks to understand how students can learn mathematics by working together with peers (e.g., Cohen, 1994b; Esmonde, 2009a; Webb, 1989, 1991). By operationalizing students' positioning practices, I have identified ways in which students' acts of positioning can bridge the gap between group or pair work and collaborative learning in mathematics classrooms. This study also indicates how students' work at the computer provides a similar, but distinct, setting in which students must manage their interactions to achieve collaborative learning.

There are practical implications from this study for the work of teaching and learning around computers in mathematics classrooms. Building upon norms for group work in mathematics classrooms (e.g., Chapin, O'Connor, & Anderson, 2003; Cohen, 1994a; Horn,

2012), teachers must be prepared to support students to collaborate around shared technology resources. For teachers to implement pair work in settings where students work at the computer, they need to manage multiple dimensions of students' work, including work on the mathematics and work on the computer. It is important to establish norms for interaction that take into account how students position themselves relative to one another through both knowledge and action moves, as well as how students are physically positioned at the computer. Moves to ask questions, request actions, and challenge peers should be capitalized upon as ways for students to contribute meaningfully to mathematical learning. Teachers should be provided with opportunities to understand how computers change the nature of collaboration and learning, so that they can support students to benefit from this work.

CHAPTER 5:

CONCLUSION

My overarching research question in this dissertation has been the following: *How do students build understanding of sine and cosine functions through working in pairs with a computer-programming environment in Algebra 2?* To answer this question, I have situated the dissertation around three interrelated strands of research (Figure 5.1). First, I studied how students invoked different conceptions of sine and cosine functions through their work on an open-ended problem with the use of Etoys. I used the cKc framework to identify conceptions in students' work, giving specific attention to how students used the tools of Etoys in their conceptions. Second, I considered students' learning about sine and cosine functions through both quantitative and qualitative perspectives. I used pre- and post-tests to compare measures of student learning, and I used a case-study analysis to look more closely at the ways individual students appropriated the tools of Etoys in their work. In the third strand, I examined students' positioning practices to study students' collaboration and participation in problem solving activities with their peers. I capitalized on students' discourse as a way to identify how students positioned themselves and their peers during pair work.

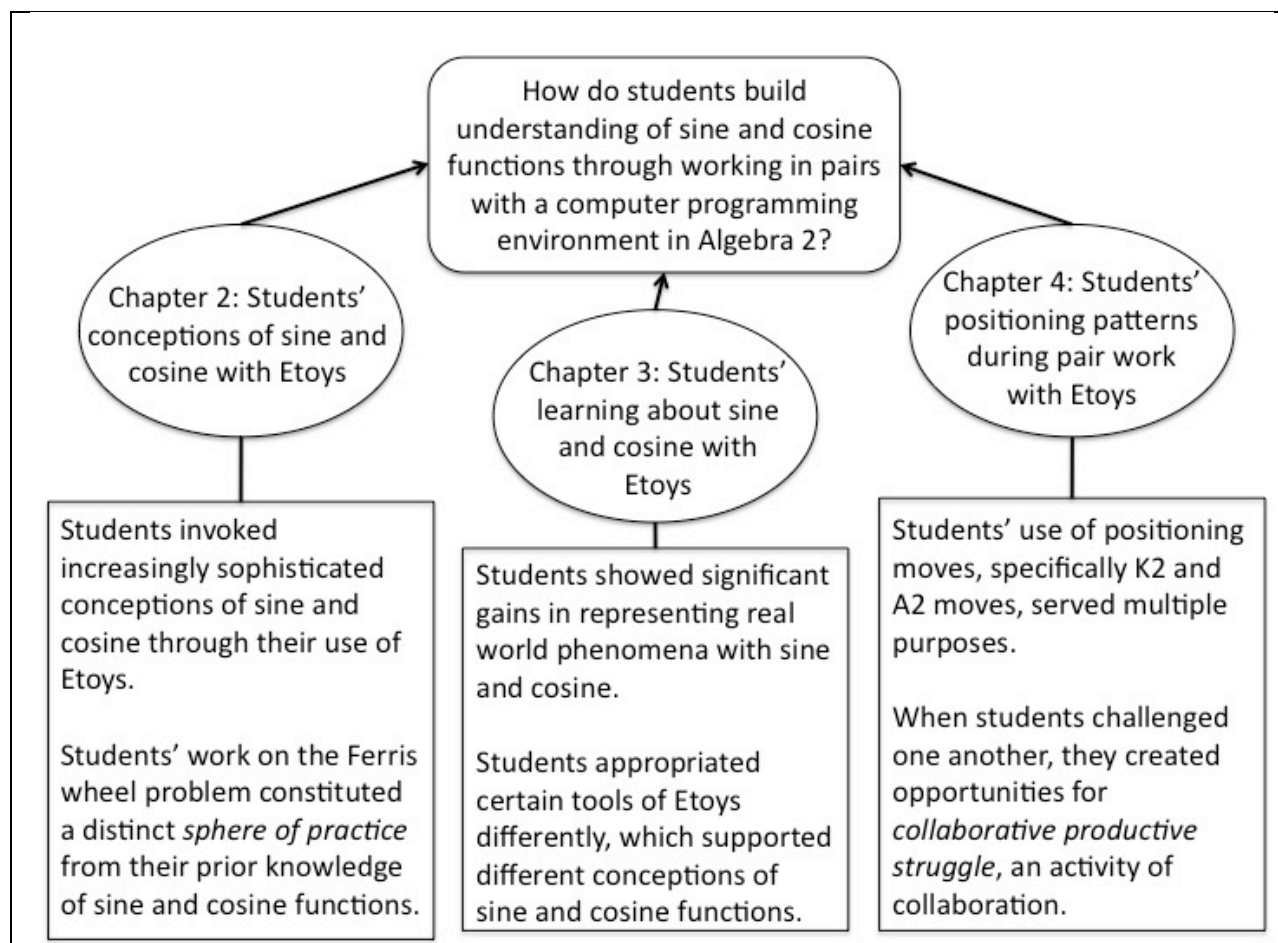


Figure 5.1. Map of three studies contributing to one overarching question.

The purpose of examining my overarching research question through this collection of studies is to benefit from the different ways that students' learning can be understood in mathematics classrooms. Cobb (1994) has suggested that there are advantages to drawing on multiple theoretical perspectives to examine students' mathematical learning. The *constructivist* perspective of learning suggests that students learn by building up units of knowledge of mathematics. At the same time, the *sociocultural* perspective identifies learning through increased participation in mathematical practices. Each of these two theories can inform the other. Moreover, for the purpose of examining practice, the benefits of combining these two perspectives are worth the price of the potential tensions between them.

By considering both constructivist and sociocultural perspectives of students' learning during the Etoys lesson, I was able to highlight the individual conceptions of students as well as the social processes through which those conceptions were established. Each of the studies in this dissertation informs the others. For example, that many students began the Etoys lesson with relatively limited prior knowledge of sine and cosine functions (Chapters 2 and 3) provides a backdrop to understand what students had to achieve through their problem solving efforts (Chapter 4). Evidence of students' challenge moves promoting collaborative productive struggle gives insight into how students' interactions about the Etoys lesson may have promoted increasingly sophisticated conceptions. Taken together, the collection of studies targets the intersection of students' collaborations with peers and their use of the Etoys environment. Below, I summarize the main findings and implications of each paper. Next, I discuss the limitations of my dissertation studies. To conclude, I outline directions for future research that will build upon and expand the work I have done for this dissertation.

Chapter 2: Students' Developing Conceptions of Sine and Cosine Functions

In Chapter 2 I asked the question, *how did students build understanding of sine and cosine functions in Algebra 2 through pair work on an open-ended problem with Etoys?* The first major claim of this study was that students' movements between conceptions of sine and cosine indicated that students were learning about sine and cosine functions through their work on the Ferris wheel problem. After identifying the conceptions that surfaced in students' work, I identified a hierarchy of those conceptions according to what was most viable for solving the Ferris wheel problem. Learning within the cK ϕ framework is defined as a process which allows an individual to move from one conception to another (Balacheff & Gaudin, 2003). By identifying a hierarchy in students' conceptions, I argued that their learning was evidenced not

only by moving laterally between conceptions but also by moving forward towards increasingly sophisticated conceptions. Moreover, as students moved back and forth between intermediate conceptions of sine and cosine, their activities with Etoys provoked them to develop more sophisticated conceptions that would allow them to solve the problem.

Students' appropriation of the tools in Etoys was integral to the operations, representations, and control structures in their conceptions. I identified that students used the scanning, drag-and-drop, and x -increase-by tools in ways that provoked them to think *against* their previous conceptions (Herbst, 2005). Specifically with the drag-and-drop tool, students had to use Etoys in a way that met the syntactical requirements of the environment. Because of this, students could not rely entirely on symbolic conceptions of sine and cosine to solve the problem. Students' learning about the tools of Etoys was an inherent part of their learning about sine and cosine functions. One could suggest that, given students' prior knowledge about trigonometric functions, their main hindrance towards solving the Ferris wheel problem may have been learning how to use the tools of Etoys. It is unlikely that Etoys was the only hindrance in students' work towards solving the problem, especially because Ms. Alexander suggested to students that they could solve the problem directly on paper. It seems more likely that students developed their understanding of sine and cosine functions as they appropriated the tools of Etoys for their mathematical work.

My second major claim in this chapter was related to students' prior knowledge about sine and cosine functions. Specifically, even though students' work on the Ferris wheel problem came at the end of a unit on sine and cosine functions, students did not seem to transfer their prior knowledge of these functions to this new setting. Once students recognized that they would use a sinusoidal function to represent the Ferris wheel ride, the Ferris wheel problem could have

been seen a straightforward computational problem about sine and cosine functions (see Figure 5.2). Although the two problems in Figure 5.2 are isomorphic to an expert observer, for students the Ferris wheel problem did register as directly connected to prior work they had done. It seems that the Ferris wheel problem constituted an entirely new *sphere of practice* (Bourdieu, 1990) for students than the previous trigonometric problems they had worked on.

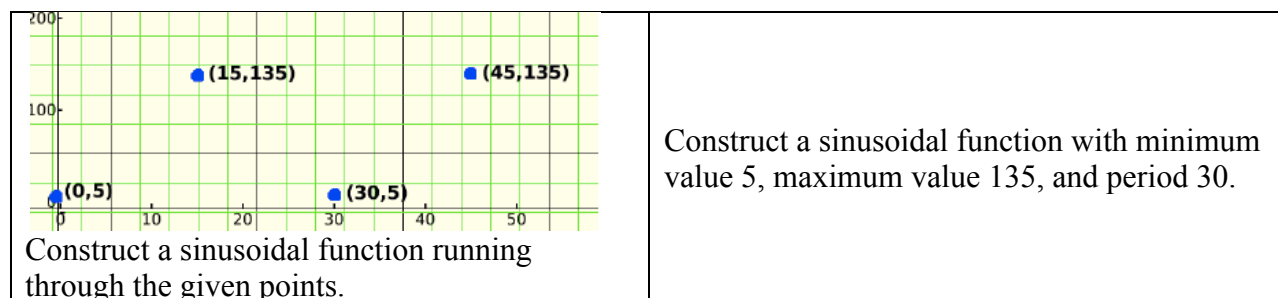


Figure 5.2. A comparison of the Ferris wheel problem, on the left, and a non-contextual problem with the same mathematical structure, on the right.

An important implication of this is that mathematics problems, which seem the same to an expert observer, are not obviously the same from the perspective of students. This is not an entirely new finding. Within school mathematics, students often invoke conceptions of mathematical concepts in certain settings that are not viable in other settings (e.g., Martínez-Planell et al., 2012; Miyakawa, 2004). The cKc framework, and the notion of spheres of practice, adds a layer of insight into why this may be the case. Students may distinguish between different spheres of practice in order to accommodate conflicting conceptions. In other words, it may not be that students forget previously learned knowledge, or that they simply do not see connections between different contexts, but rather that students compartmentalize different experiences so as to maintain viable conceptions in different settings. Especially in light of the many different contexts in which students study trigonometric ratios and functions, it is critical to support students to make connections between the different domains of validity in which they

invoke different conceptions. By bridging the gaps between students' spheres of practice, instruction can help students replace overly simplistic, or non-viable, conceptions with more sophisticated ones.

Chapter 3: Student Learning About Sine and Cosine Functions

In Chapter 3 I asked, *what evidence did students show of learning about sine and cosine functions through their work with Etoys?* Two major claims emerged from my work in this chapter. First, comparisons of students' pre- and post-tests indicated that students' scores on a post-test item about representing real-world situations with trigonometric functions were significantly better than their performance on the pre-test item. This finding revealed the potential for students to learn through work on an open-ended problem in a way that will transfer to new problem solving situations. Beyond that, implications from the quantitative analysis of students' pre- and post-tests were limited, given that many of the differences in scores were not statistically significant. However, the limitations of that work raised important points about the nature of students' prior knowledge that will appropriately determine students' potential to benefit from computer-programming activities. For example, it may be the case that students who specifically struggle with graphical representations would benefit the most from the dynamic representations offered by a programming environment such as Etoys. Future measures that offer a more nuanced view of student thinking and understanding may be better indicators for when and how students are likely to benefit from the use of a programming environment.

The qualitative portion of this chapter gave case studies of five different students who solved a problem analogous to the Ferris wheel problem. From this, it became clear that some students used Etoys in ways that supported increasingly sophisticated conceptions, while other students found ways to accommodate the Etoys syntax with limited conceptions of sine and

cosine. Although all students had the tools of Etoys at their disposal, students appropriated the tools of Etoys in different ways, which led to different sorts of instrumented activity. Identifying the tools available in programming environments is one step towards building the research base on how students can use these environments to learn mathematics. However, it is not sufficient to know which tools students use. It is necessary to know how students use different tools within a programming environment and how a student's use of those tools is part of a broader mathematical conception. Moreover, it is important to consider how tasks with technology, and technology tools themselves, are designed to provoke students to use tools in mathematically productive ways.

There are implications of this second claim regarding the design and implementation of mathematics tasks with computer-programming environments, as well as the design of programming environments themselves. The introduction of technology tools for mathematics learning is not useful if those technology tools only lead to slight variations of overly simplistic or limited conceptions. Ideally, students who engage in a programming activity with limited conceptions should emerge from that activity with improved conceptions. Students' activities should become instrumented in ways that support mathematical learning. However, this goal means that mathematical tasks need to be designed so that students' use of programming tools will challenge their initial conceptions. To design such a task, researchers and practitioners need to be cognizant of students' different conceptions of a mathematical concept, and they need to have a road map in mind for how students will reach new understandings. The construct of the *hypothetical learning trajectory* (Gravemeijer, 2004; Simon, 1995) offers a way to define learning goals and activities by taking into account students' evolving thinking about specific concepts. Learning trajectories can provide a useful lens for teachers to reflect on the work of

teaching, through their consideration of paths of student learning (Wilson, Sztajn, Edgington, & Confrey, 2013). The hierarchy of conceptions of sine and cosine that I established in Chapter 2 offers a small-scale trajectory of students' learning about trigonometric functions. Providing teachers with resources of this nature can provide a way for teachers to identify student thinking and to predict how students may progress in their thinking. This information is critical in order to make the value of programming activities, in terms of student learning, worth the time investment to engage students in these activities.

Chapter 4: Students' Positioning Patterns During Pair Work With Etoys

In Chapter 4 I asked the question, *how did pairs of students' patterns of positioning support or inhibit their collective problem solving efforts?* The first major finding of this work was that students' positioning moves served multiple purposes. This was especially true in the performance of K2 moves. By performing K2 moves, students positioned themselves as novices and thereby positioned their partners as experts. But more importantly, performing K2 moves allowed students to put ideas on the table for discussion. Moreover, students' performance of K2 moves created opportunities for discussion within the pair. Students who acted as novices during work in partners promoted mathematical discussions. This finding suggests that categorizing students according to who is the expert and who is the novice may be an overly simplistic distinction. Identifying a student as a novice in a pair or group may hold negative connotations, for example that the student does not contribute mathematically as much as the expert. The smooth functioning of pairs and groups of students requires that students are willing to ask each other questions and suggest ideas for the other members of the group to consider (Barron, 2000, 2003; Chizhik, 2001; Ellis, 2011). This chapter suggested that there are different ways a student can act as a novice, and it is not necessarily the case that the novice participates less

meaningfully than the expert. The novice in a pair or group of students may in fact be the person who pushes the conversation forward.

The second main claim of this chapter comes from establishing the construct of *collaborative productive struggle*, drawing on Hiebert and Grouws's (2007) notion of productive struggle. An instance of collaborative productive struggle occurs following a move by a student to challenge his or her peer regarding the mathematical content at hand. Following that challenge move, both students in a pair contribute K2 or K1 moves regarding that mathematical content until the issue is eventually resolved in a way that makes progress towards a correct solution to the problem. Collaborative productive struggle is collaborative in the sense that both students contribute meaningfully, by either asking questions or offering ideas, to the discussion. It is a struggle, because the collaboration stems from a disagreement, which is evidenced by the challenge move. Finally, the struggle is productive in that it leads to desirable mathematical outcomes from the perspective of an expert observer.

When students decide to challenge their peers during pair work, they also make choices about how to respond to those challenges. Students must choose whether they will react to a challenge move or ignore it. If they react, the discussion that follows could either resolve the content of the challenge or eventually abandon it. The choices that students make regarding how they respond to challenges from their peers are not, in general, made explicit. But the results of those choices mean that some occasions of challenge moves provoke students to engage in collaborative problem solving efforts. Collaborative problem solving efforts have both mathematical and interpersonal implications for students. Mathematically, students take a step further towards solving a problem they did not know the answer to. Interpersonally, students have a shared ownership of the solution. Since collaboration cannot be assumed as a result of

assigning students to work together (Staples, 2007), it is important to recognize instances when students pursue opportunities that lead to collaboration. The more those opportunities can be made explicit in research and practice, the more that teachers and students can capitalize on them.

The claims of this chapter have implications for how teachers can establish and support norms for interaction between students during group and pair work. Using students' discourse to examine collaborative productive struggle is a new strand of research, and there is a great deal of work to really answer how teaching actions correlate with the occurrence of collaborative productive struggle between students. Even from the outcomes of this study, however, implications can be gleaned regarding the appropriate norms for students to engage in conversations with their peers. First, there is an implication for students that *asking questions can promote mathematical discussions*. This suggestion is reminiscent of Webb's (1989, 1991) findings regarding the importance of asking questions during group work. While Webb emphasized the importance of asking questions for lower-achieving students to improve their mathematical performance, this study takes the importance of asking questions a step further. Not only do questions support the learning of the novice, but also asking questions can support the problem solving efforts and collaboration of the group as a whole. This is critical for understanding that asking questions is a valuable activity for all members of a group. Teachers and students may benefit from establishing guidelines for *what* sorts of questions to ask, *how* to pose a question, and *when* is the appropriate time to ask questions. These guidelines will contribute to practices for teaching students how to interact and disagree (e.g., Cohen, 1994b; Horn, 2012).

Another implication from this study is that *challenging your peers' mathematical ideas creates opportunities for joint mathematical understanding*. Research on the ways that students challenge one another's ideas in mathematics classrooms has been somewhat inconclusive. In some cases, students who are challenged may stop contributing to the group (Johnson & Johnson, 1985; Watson & Chick, 2011). In other cases, overly tenacious students may ignore challenges from their peers, which makes the challenge moves irrelevant (Barron, 2000; Watson & Chick, 2011). With these different findings, it is important to establish clear guidelines for the appropriate ways for students to talk to one another, and what it really means to challenge the mathematical content of someone's talk. In the practice of mathematics, disagreements are critical opportunities to refine previous conjectures and establish new understandings (Lakatos, 1976). The importance of students learning to disagree about mathematical ideas is made apparent in current standards documents, which suggest that students should be able to, "construct viable arguments and critique the reasoning of others" (NGAC, 2010, p. 6). Even though students' challenge moves are not always productive during group work, it is a priority within the mathematics education community for students to learn how to respond to mathematical arguments. Teachers can support this endeavor by promoting appropriate ways for students to challenge peers and respond to those challenges.

Limitations of the Study

The studies comprising this dissertation have three main limitations. First the setting of the studies, with secondary students working on a programming activity to learn mathematics, is not representative of a typical mathematics classroom. For that reason, my analysis of students' work in this setting may be characterized as an *investigation of possibilities* (Stylianides, 2005) rather than an examination of the typical work that students do in school mathematics. In other

words, this dissertation offers insight into the potential for students to learn mathematics through a computer-programming activity. Research on the use of programming environments is limited, and the research that does exist is often situated with students younger than high school and not necessarily learning mathematics concepts (e.g., Fessakis, Gouli, & Mavroudi, 2013; Fujioka, Takada, & Hajime, 2006; Kelleher, 2006; Valente & Osorio, 2008). However, studying students' mathematical learning through programming activities shows promise for at least two reasons. First, there is precedent for this type of work, particularly with the various Logo environments that were particularly prevalent in mathematics classrooms at the end of the 20th century (e.g., Clements, Battista, & Sarama, 2001; Edwards, 1997; Hoyles & Healy, 1997). Although, again, much of this research was conducted with elementary and middle grades students, it is a natural extension to ask how students in later grades can benefit from programming activities. In addition, research in computer science education has been making strides in understanding how computer programming can support *computational thinking* (Wing, 2006), a construct encompassing many of the same practices and habits of mind that are desirable for students learning mathematics. In the following section, I will discuss the potential to make connections with the research base in computer science education and expand my own research in mathematics education using computer-programming environments.

The second major limitation of my dissertation is that I have conducted my studies within three different classrooms all taught by a single teacher in a single school. To examine students' conceptions of mathematics, it would be ideal to work with multiple classrooms, taught by different teachers, in different school contexts. Such a design would allow for comparisons based on students' prior knowledge and prior experiences using technology in mathematics classrooms, and based on institutional factors such as whether a school employs tracking and the

different curricula that teachers use. To study students' discourse when collaborating with peers, it would be advantageous to be able to compare students in schools with different levels of diversity, and classrooms with different norms in place for group work. Students at Grove High School were accustomed to working on "rich tasks" in Ms. Alexander's classes, which supported a classroom norm of students working in groups with peers to solve mathematics problems. However, Ms. Alexander did not have specific norms in place for *how* students should talk with one another, for example the norms established through Complex Instruction (Cohen, 1994a, 1994b).

Even though it would be ideal to be able to establish comparisons through a study design with multiple sites, there is still a great deal to learn from a close examination of one teacher's Algebra 2 classes. Primarily this is due to the fact that there is little research of the same nature as this dissertation. Given that this work is somewhat exploratory, and largely descriptive, it has been advantageous to hold some variables constant, for example classroom expectations for group work at the time of the lesson. A natural question that emerges from the construct of collaborative productive struggle is how teachers can establish a link between classroom norms for group or pair work and students' interactions to lead to collaborative productive struggle. I will examine this question, and others that emerge from the possibilities of expanding this research to compare students in diverse schools and classrooms, further in the following section.

Finally, with this body of work, I have considered students' mathematical learning apart from the social, cultural, and individual factors that contribute to student learning. Student thinking and learning do not exist apart from students' race, culture, socioeconomic status, and identity in mathematics classrooms (Esmonde & Langer-Osuna, 2013; Gutiérrez, 2013; Lubienski, 2000; Nasir, Hand, & Taylor, 2008; Zevenbergen, 2000). In mathematics education

research, examinations of student conceptions and thinking often come without attention to how those conceptions are shaped by students' non-mathematical experiences. However, a critical part of the work of researchers examining issues of student learning is to ask critical questions about which students have access to certain learning opportunities, and how students' learning experiences are shaped by their personal identities (Nasir, 2013). Examples of this type of work exist, for example the studies of Boaler and colleagues, which combine in-depth examinations of students' work in classrooms with quantitative measures of learning, and do so in light of students' diverse backgrounds (Boaler, 2002, 2008; Boaler & Staples, 2008). Such work should serve as an example to me in my future research. Specifically, I aim to further articulate frameworks for understanding students' conceptions and participation in group discourse in ways that prioritize students' backgrounds, language, schooling experiences, and access to mathematical learning.

Directions for Future Research

In my work thus far I established a research base with two major components: students' use of programming environments and students' collaboration for learning mathematics. I have specifically considered students' learning, and I am interested to pursue broader questions about how mathematics instruction is implemented within these contexts. Cohen, Raudenbush, and Ball (2003) proposed a model of instruction as a triad, composed of interactions between teachers, students, and mathematical content (see also Kilpatrick, Swafford, & Findell, 2001). Within this triad, "teaching is what *teachers* do, say, and think with *learners*, concerning *content*, in particular organizations and other environments" (Cohen et al., 2003, p. 124, emphasis added). Teachers, students, and content play interdependent roles in the teaching and learning that happens in mathematics classrooms. To guide my future research, I have identified questions

according to these three components of the triad (Figure 5.3). I begin with questions related to students' learning through interactions with peers and the use of technology tools. Second, I consider a question about the design of tasks using technology to teach mathematics. The question of task design targets one avenue through which teachers and students engage with mathematical content. Finally, I consider questions about the work of teaching in settings where students learn mathematics with technology through pair and group work.

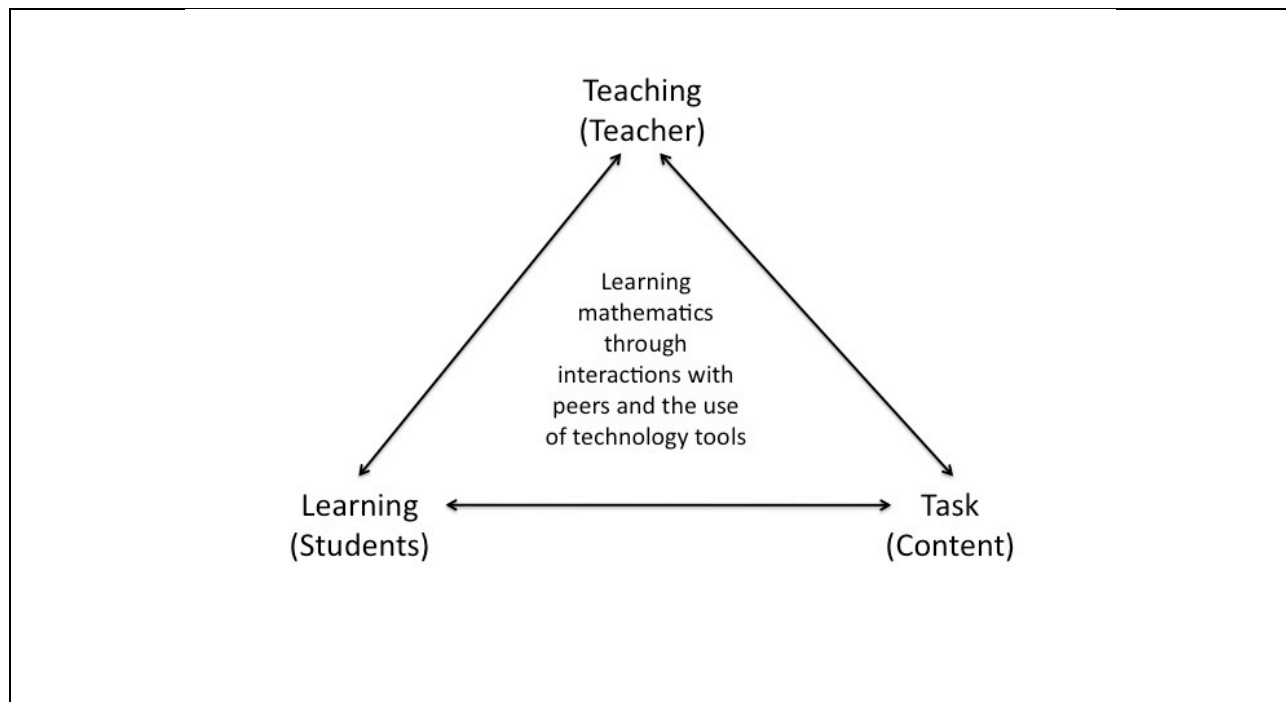


Figure 5.3. A model for future research, based on the model of the instructional triangle (Cohen, Raudenbush, & Ball, 2003; Kilpatrick, Swafford, & Findell, 2001).

Research on Learning

My first question seeks to broaden my study, beyond the concept of sine and cosine functions, of how students' use of programming environments support mathematical learning: *In what ways do students' conversations around computer-programming environments strengthen connections between procedural fluency and conceptual understanding in mathematics?*

Kilpatrick, et al. (2001) defined *mathematical proficiency* as being made up of five interrelated

strands: conceptual understanding, procedural fluency, adaptive reasoning, strategic competence, and productive disposition. All of these strands should exist in tandem, and competency in one should support the development of the others. However, mathematics education research has often worked from an assumption that procedural knowledge is superficial and without connections, while conceptual understanding is extensive and rich in relationships (Star, 2007). Moreover, especially in the study of Algebra, students often focus on the mastery of procedures at the expense of conceptual understanding (Chazan et al., 2007). There is potential, however, for students to develop procedural knowledge that is itself deep and interconnected, and also connected to conceptual understanding (e.g., Star & Rittle-Johnson, 2008). My studies of students' use of Etoys to learn about sine and cosine functions established that students developed increasingly sophisticated conceptual understanding of these trigonometric functions while at the same time formulating and refining procedures to develop a solution to the problem. These findings were very specific to the mathematical content, but they suggest the potential for students' use of programming environments to support connections between procedural knowledge and conceptual understanding.

Students' work with Etoys was critical for supporting their conceptual understanding and their procedural work. Prior research has identified that students' use of technology tools enables students to create and talk about mathematics in new ways (Healy & Hoyles, 2001; Hoyles & Noss, 1992). I expect that, as students use the tools of a programming environment to accomplish certain procedures, their conversations about the use of those tools will support their conceptual understanding about those procedures. This was often the case in this dissertation. For example, as students talked about how to use the drag-and-drop tool to construct their functions, they began to make sense of the meaning behind the various components of the

function. One of the greatest advantages of a programming environment is that it allows users to maintain symbolic control of their work (Healy & Hoyles, 2001). I am interested to examine how students' use of programming environments can support procedural fluency that is rich and connected, and that is also linked to conceptual understanding.

There are many topics in Algebra that are rich in their conceptual underpinnings, but which also require substantial procedural mastery. Sine and cosine functions are one example of many families of functions that are important in secondary mathematics and beyond, including all trigonometric functions, exponential functions, and power functions. Managing the symbolic representations of functions requires procedural fluency, but to understand the meaning behind the symbols requires conceptual understanding of the meaning of function and the relationships between quantities. Other topics that include a high degree of procedural fluency include rationalizing expressions, solving systems of equations, and establishing polynomial and trigonometric identities. Mathematical modeling activities, which with current standards have an increasingly important role in school mathematics (NGAC, 2010), require proficiency with managing data, selecting an appropriate function to model, and selecting the appropriate layout for presentation of data. All of these areas of mathematics require procedural fluency, which should not come at the expense of conceptual understanding. In my future work, there is a great deal of potential to consider how computer-programming environments can establish links between these two strands of mathematical proficiency, especially in areas that are often largely procedural from the perspective of students.

My next question explores connections between students' mathematical learning and the type of thinking provoked by a computer-programming environment: *How can computational thinking support students' mathematical proficiency?* This question connects mathematics

education research with research in computer science education. Computer science education has defined a construct of *computational thinking*, which refers to a process of conceptualizing problems, coming up with ideas to solve them, and combining mathematical and engineering thinking to solve problems (Wing, 2006). Computational thinking encompasses many of the mathematical practices that are identified in the *Common Core Standards for Mathematics* as areas in which students should develop expertise, including making sense of problems, reasoning abstractly, and using the appropriate tools (NGAC, 2010). Computational thinking places a priority on understanding and using the increasingly robust tools that are available for problem solving.

As students use programming environments to study mathematics, I expect that their computational thinking will increase through their use of those environments. Based on that expectation, I intend to examine whether students' computational thinking will support students towards developing greater mathematical proficiency. Given that computational thinking inherently draws upon mathematical thinking (Wing, 2006), it is reasonable to expect that each will support the development of the other. As computer environments for learning mathematics become more robust, it will be important to know whether and how the ways that students conceptualize and solve problems around the computer support mathematical learning. This knowledge will have implications for how tasks are designed, as well as when and how it is appropriate to introduce programming environments in mathematics.

Beyond my examinations of students' mathematical thinking, I am interested in exploring the factors that shape students' experiences working in pairs around computers: *What individual factors impact students' opportunities to position themselves during pair work around the computer?* Individual level factors include things such as students' race, socioeconomic status,

and students' social status among their peers. With even the best intentions of engaging all students in meaningful mathematical practices, not all students benefit in the same way from certain classroom activities. Students who are non-dominant members of the classroom community, according to socioeconomic status, race, or gender, may be marginalized in discussion intensive, open-ended problem solving settings (Lubienski, 2000; Murrell, 1994; Webb, 1984; Zevenbergen, 2000). Even among relatively homogeneous groups of students, issues of status can arise according to which students are more popular or which students are perceived as the smart students in the class (Cohen, 1994a). When students work together around computers, there is potential for a new status issue to arise, specifically according to which students are assumed to be the most proficient with technology. In previous research, students who have been identified as good readers among their peers have assumed higher status during group work activities, even if those activities did not depend on reading ability (Rosenholtz, 1985). It may be the case that students who are identified as more computer savvy by their peers will assume positions of authority during work around computers, even if students identify themselves equally in terms of mathematical understanding. For students' interactions around computers to be productive, it is important to identify ways in which students' opportunities to position themselves are influenced by their status among students in the class.

Finally, I intend to continue my work to attempt to identify patterns of positioning that correlate with instances of collaborative productive struggle: *What positioning moves, or patterns of positioning moves, either support or inhibit students' participation in collaborative productive struggle?* The construct of collaborative productive struggle is useful in that it identifies instances of collaboration in students' work. An open question remains about whether certain acts of positioning are more likely to promote collaborative productive struggle between

students. In Chapter 4 I focused specifically on students' challenge moves regarding mathematical content. But a finer grained analysis of those moves may reveal that different ways of challenging students' mathematical ideas can be more or less productive. For example, the level of specificity that students use in their moves to challenge peers, or the level of emphasis given to a challenge, may affect the ways that students respond. In a study of middle school students, I have begun to identify that students perform a variety of discursive moves, including action moves that serve to challenge their peers (DeJarnette, 2013). Giving a more thorough description of how different moves either support or inhibit collaboration between students will contribute to existing norms for group work that will be useful and productive for teachers and students in classrooms.

Research on Task Design

A question about the design of tasks with programming environments stemmed from my observations of the different ways that students benefited from their use of Etoys: *What features in the design of a task using technology tools will challenge students' initial conceptions and promote new understanding?* This question stems largely from the differences in students' use of the tools of Etoys, which were revealed during the post-lesson interviews. While some students appropriated the tools in ways that elicited improved conceptual understanding, other students learned to accommodate the tools of Etoys to satisfy limited understandings. This finding is one of a number of possible pitfalls of students' use of technology environments, including programming environments but also extending to students' use of calculators, computer algebra systems, and dynamic geometry environments. There are instances when students may rely too much on immediate feedback from the computer, may forego productive strategies to accommodate the tools of an environment, or may use technology tools to confirm

or check ideas rather than to devise new solution strategies (e.g., Healy & Hoyles, 2001; Hollebrands, 2007; Hölzl, 2001; Yerushalmy, 2006). With knowledge of these pitfalls, there is potential to consider how they might be avoided through the design and implementation of mathematical tasks.

In the case of programming environments, I am interested to examine the question about how a mathematical task should be designed to fully leverage the potential of the environment to support students' learning. I expect it will be important to consider what questions a task poses for students, and specifically what types of proficiency (e.g., procedural fluency, conceptual understanding, strategic competence) those questions are designed to foster. In addition, the way the available tools are designed and presented to students within the technology environment is integral to the design of the task. There may also be benefit from tasks that require students to engage in some metacognitive thinking about how they are using the available tools to solve the problem. By tweaking different elements in the design of a mathematical task, I intend to contribute to a stronger understanding for how tasks with technology can be designed to support students' mathematical learning.

Research on Teaching

Concerning the work of teaching, I intend to examine how a teacher's actions support students' collaborations: *How can teachers establish norms for classroom interactions that support students to engage in collaborative productive struggle?* For students to engage in productive collaborations in mathematics, there is a great responsibility on the teacher to support these collaborations. As I mentioned in the earlier section, implications have emerged from my study regarding norms that teachers may establish for pair work around computers, including a norm for asking questions and norms for how to challenge peers. Resources for practitioners

have provided a great deal of information about how to engage and manage students' mathematical discussions as a whole class and in groups (Chapin, O'Connor, & Anderson, 2003; Herbel-Eisenmann & Cirillo, 2009; Horn, 2012; Smith & Stein, 2011). These resources outline how to design and launch a task that will encourage collaboration, how to support students to communicate mathematical ideas, and how to manage interpersonal relationships during group work. Research in mathematics education has benefited from Cohen's work on Complex Instruction (Cohen, 1994b; Cohen & Lotan, 1995), an instructional technique that is specifically designed to manage status differences among students. The norms and instructional techniques that teachers implement can be considered through the lens of how they impact students' positioning during pair and group work.

I intend to examine how teachers can establish and implement classroom norms in ways that support students to position themselves equitably towards one another during group work and pair work. By positioning themselves equitably, I mean that students distribute K1, K2, and challenge moves between them so that no single student is always positioned as the expert within a pair or group. For example, as teachers set norms about asking questions, I expect more students would be inclined to perform K2 moves, so that students trade the position of novice. By implementing norms for how students should challenge and respond to one another's challenges, I expect that teachers can support students to engage in more instances of collaborative productive struggle. I also intend to examine how classroom norms for collaboration may be tweaked to accommodate settings where students are working together at a computer. It may be the case, for example, that the ways students should challenge one another and respond to those challenges have unique features in settings where physical control of the computer is up for negotiation. As students increasingly work together around technology tools,

it will be important to identify how norms for group work in mathematics classrooms will support students to work together with technology.

Finally, I intend to study how teacher learning about the role of technology in teaching mathematics can support student learning: *How does a teacher's knowledge of technology tools correlate with student learning outcomes?* In addition to examining the different ways that students learn mathematics through interactions around technology tools, it is important to establish connections between the ways teachers teach with technology and what students learn. The formulation of *Technological Pedagogical Content Knowledge* (Mishra & Koehler, 2006; Niess, 2005), or TPACK, builds on Shulman's (1986, 1987) idea of pedagogical content knowledge and extends it to consider how teachers integrate technology into their teaching. Teachers must manage content and pedagogy at the same time as technology. Niess (2005) identified four components for fostering teachers' TPACK. First, teachers need an overarching understanding of what it means to teach a subject, in this case mathematics, with technology. Second, teachers need knowledge of instructional strategies for teaching certain topics with certain technology tools. Third, teachers need to have knowledge of students' understanding and learning with technology. Fourth, teachers need knowledge of curriculum materials that integrate technology into students' learning. The second and third points seem especially crucial for understanding how a teacher's work in the classroom shapes students' learning of mathematics with the use of technology. The strategies that teachers use, and their knowledge of student thinking, are critical for linking a teacher's work to student learning outcomes.

With my dissertation studies, I have considered students' learning with Etoys largely apart from the work of the teacher. This choice allowed me to give priority to student interactions. Moreover, since teachers do not have direct access to students' mathematical

understandings (von Glasersfeld, 1993), there is benefit to looking directly at student work rather than how that work is filtered through conversations with a teacher. However, with the basis of knowledge I now have about students' learning through their use of Etoys, I am better equipped to consider questions about how the work of the teacher shapes this learning. Instruments exist for using the TPACK framework as a tool for classroom observations of teaching (Koehler, Mishra, Yahya, & Yadav, 2004) and through surveys with teachers (Koehler & Mishra, 2005). In addition, Wilson, Lee, and Hollebrands (2011) have identified different categories of ways that pre-service teachers pay attention to student thinking with technology. By drawing on the research base of existing models and instruments for examining TPACK, I expect to begin to establish a relationship between teachers' technological pedagogical content knowledge and students' mathematical learning.

Concluding Remarks

I began this project with an interest in how secondary students could learn mathematics through pair work with the use of a computer-programming environment. Based on my own experiences with Etoys, and my own appreciation for the complexities of mathematics, I saw potential in Etoys to provide students new ways to engage in mathematical thought and problem solving. Although students in this study did not, in general, have prior programming experience, they found ways to use the tools provided in the Etoys environment. Through their interactions with peers, students learned to manage the tools of Etoys. As that learning occurred, students also gained mathematical insights that helped them solve the problem.

Mathematical programming languages have changed what is possible to do in mathematics and what sorts of mathematics can be expressed. At the same time, students' use of computers offer insights into students' conceptions and practices (Noss & Hoyles, 1996). By

examining students' work in partners with the use of a programming environment, I gained insight into the language by which students communicate mathematical ideas. This language was shaped by students' use of Etoys, and it revealed how students' mathematical conceptions and practices were shaped by their use of Etoys. This dissertation has revealed some of the ways that students' mathematical activity can become instrumented through their use of technology tools. With the questions this work has answered, it has also provoked new questions regarding the potential impact of computers, collaboration, and the intersection of these activities for students' learning.

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APPENDIX A:

THE FERRIS WHEEL PROBLEM

The London Eye: One of the most famous Ferris wheels in the world is the London Eye in London, England. Assume that the London Eye has a diameter of 130 meters, and the lowest point on the Ferris wheel is 5 meters above the Thames River. It takes 30 minutes to make one complete revolution. You and your partner are going to ride the Ferris wheel. You get on the Ferris wheel at the very lowest point.

1. On page 1 of your Etoys notebook is a model of the London Eye Ferris Wheel. Experiment with that model, and make some conjectures about how you could model your height off the ground as a function of time. Write your ideas in the space below.
2. On page 2 of your Etoys notebook is an example of how to write a script to make a graph. Explore with that script to get comfortable with the commands. The green dot is your “plotter.”
3. On page 3 of your Etoys notebook there is an empty graph, with some menu options for you to write a script of your own. First, find and plot at least 4 or 5 points that model your height off the ground at a time t . Then, write a script that will model your height off the ground as a function of the time. The pink dot is your “plotter.”

5. In the space below, write a paragraph to explain the different parts of your function from #4. Make sure you describe how the numbers in your function are related to the context of the Ferris wheel.

APPENDIX B:

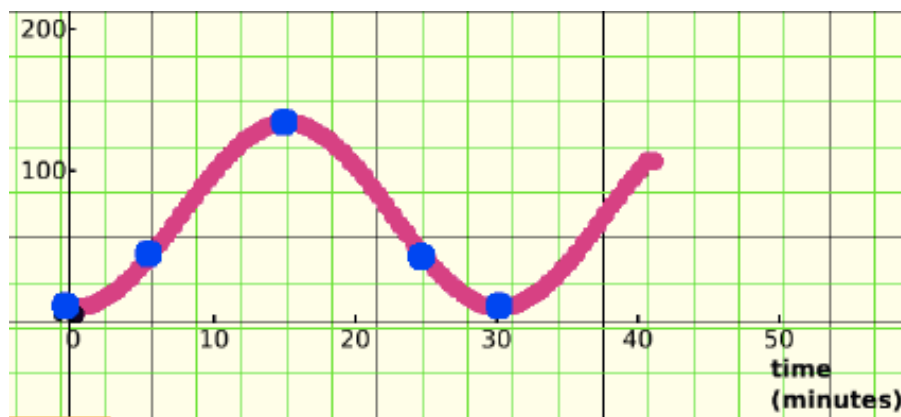
A SOLUTION TO THE FERRIS WHEEL PROBLEM

(30 MINUTES AND 130 METERS IN DIAMETER)

At 0 minutes, a person riding the Ferris wheel will be 5 meters off the ground. The table below indicates height off the ground at various moments while riding the Ferris wheel:

Time	Height off the ground
0 minutes	5 meters
7.5 minutes	70 meters
15 minutes	135 meters
22.5 minutes	70 meters
30 minutes	5 meters

When graphed, the height off the ground follows a sinusoidal pattern.



Since the graph starts at its minimum, the height can most efficiently be represented by a negative cosine function.

The midline of the graph occurs at $y=70$, so the function should be shifted vertically by 70 units.

The graph stretches from $y=5$ to $y=135$, 65 units above and below the midline.

Lastly, the graph takes 30 minutes to complete one period. To adjust our cosine function, which usually requires 2π to complete one period, we need to adjust the coefficient of the independent variable inside the cosine function. By dividing by 30, and multiplying by 2π , we can account for the period of 30 minutes.

Putting all the components together, a function to represent the height off the ground is given by

$$f(x) = -65 \cos\left(\frac{2\pi}{30}x\right) + 70$$

Other functions are equivalent to this cosine function, and could also be used to represent the height off the ground. Examples of functions that are equivalent include

$$f(x) = 65 \sin\left(\frac{2\pi}{30}x - \frac{\pi}{2}\right) + 70$$

and

$$f(x) = 65 \sin\left(\frac{2\pi}{30}x + \frac{3\pi}{2}\right) + 70$$

APPENDIX C:

WARM-UP PROBLEM STUDENTS SOLVED ON DAY 2 OF THE ETOYS LESSON

1. Write a cosine equation with a minimum of 10 and a maximum of 30. The period, phase shift, and vertical shift can be anything you want.
2. Write a cosine equation with a period of 15. The amplitude, phase shift, and vertical shift can be anything you want.

APPENDIX D:

TEST ITEMS FOR VERSIONS A AND B OF THE PRE- AND POST-TESTS

Version A

1A. The Music Palace is having a sale:

Music Palace Sale

\$12 for the first CD

\$6 for each additional CD

(Prices include tax.)

Write an expression that shows how to calculate the cost of buying n CDs at the sale.

2A. For the sine function $f(x) = 15 \sin(7x)$, identify the amplitude, frequency, and period.
Show all of your work.

Amplitude:

Frequency:

Period:

Amplitude: 2

Period: $\frac{2\pi}{3}$

3A. Which of the following trigonometric functions has the properties given above?

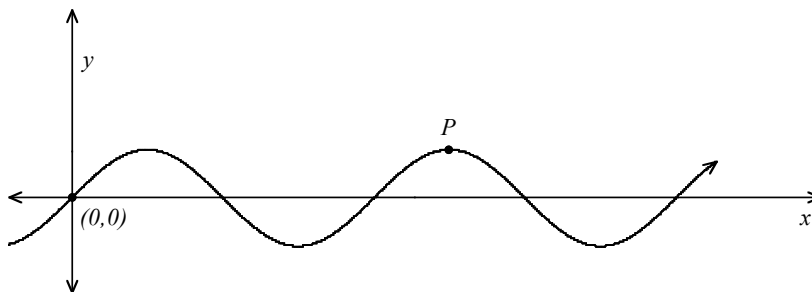
A. $y = \frac{2}{3} \cos(2x)$

B. $y = \frac{2}{3} \cos(3x)$

C. $y = \frac{3}{2} \cos(2x)$

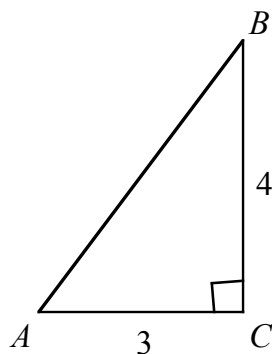
D. $y = 2 \cos\left(\frac{2}{3}x\right)$

E. $y = 2 \cos(3x)$



4A. The graph of $f(x) = \sin(x)$ is shown above. Which of the following is the x -coordinate of the point P ?

- A. $\frac{\pi}{2}$
 - B. π
 - C. $\frac{3\pi}{2}$
 - D. 2π
 - E. $\frac{5\pi}{2}$
-

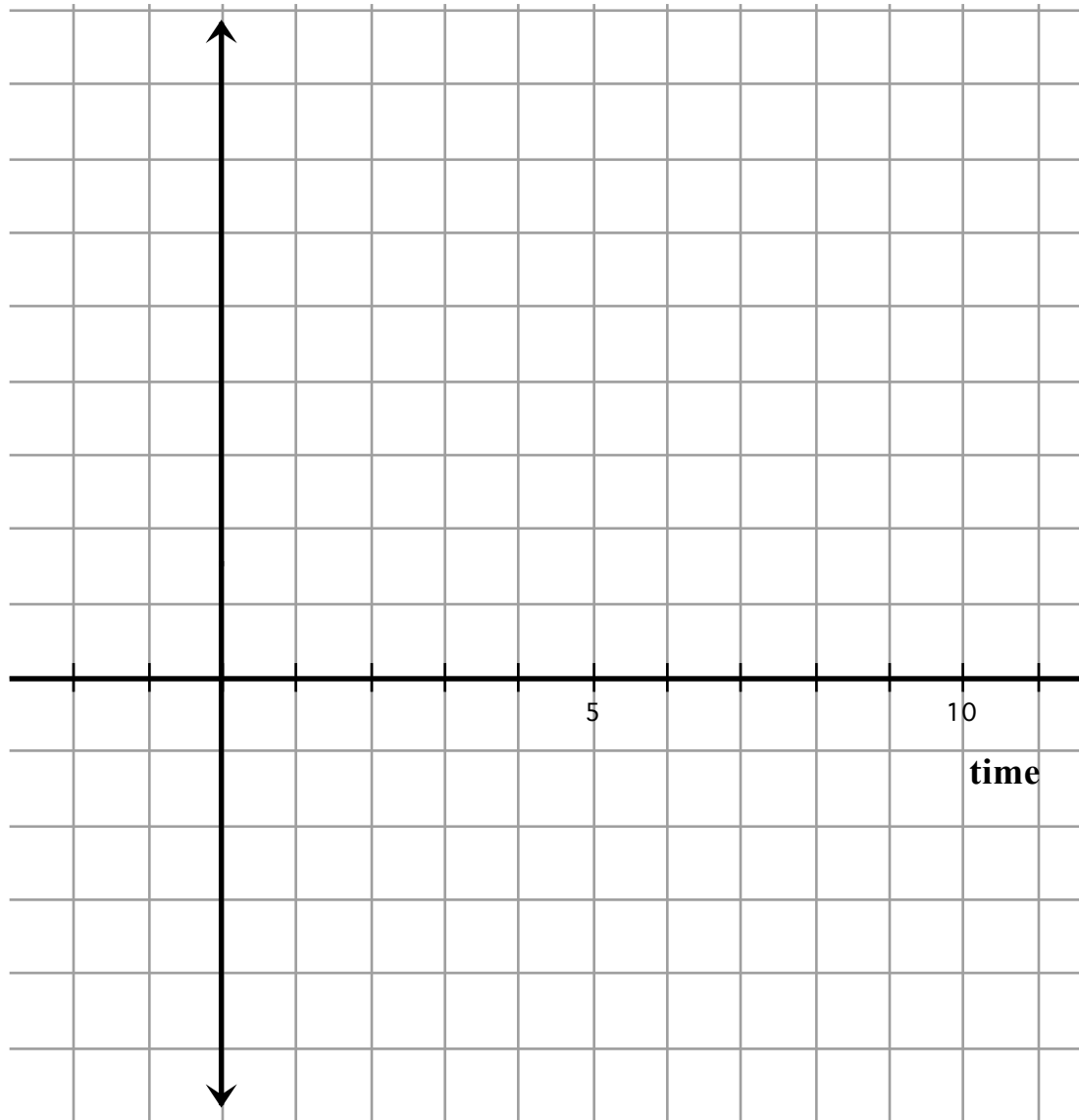


5A. In the right triangle above, $\cos(A) =$

- A. $3/5$
- B. $3/4$
- C. $4/5$
- D. $4/3$
- E. $5/3$

6A. A baby jumper is designed to let a baby sit in a secure seat that is attached to a frame and jump up and down. On one model, the seat of the baby jumper can lift the baby 60 cm off the ground, and the seat can go as low as 20 cm off the ground. It takes 1 second for the seat to cover this distance.

- A. On the axes below, sketch a graph of the baby's height off the ground as a function of time, for the first 10 seconds.



- B. Write an equation that will model the height of the baby from the floor over time.

1B. A new taxi company is advertising their cab fares:

Cab Fare

\$5 for the first mile

\$1.75 for each additional mile

(Prices include tax and tip.)

Write an expression that shows how to calculate the cost of an n mile cab ride.

2B. For the cosine function $f(x) = 10\cos(3x)$, identify the amplitude, frequency, and period. Show all of your work.

Amplitude:

Frequency:

Period:

Amplitude: 3

Period: $\frac{5\pi}{2}$

3B. Which of the following trigonometric functions has the properties given below?

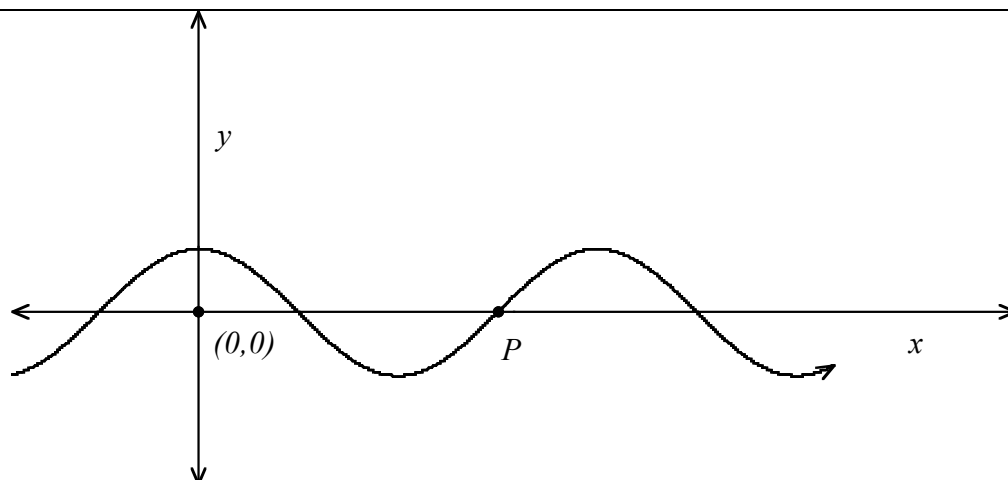
A. $y = \frac{5}{2}\sin(3x)$

B. $y = \frac{5}{2}\sin(5x)$

C. $y = \frac{2}{5}\sin(5x)$

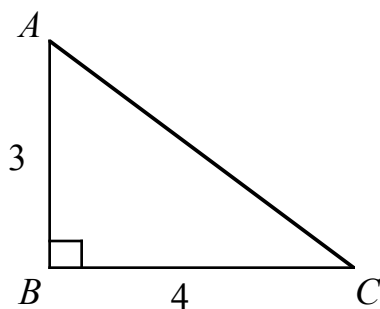
D. $y = 3\sin(\frac{4}{5}x)$

E. $y = 3\sin(5x)$



4B. The graph of $f(x) = \cos(x)$ is shown above. Which of the following is the x -coordinate of the point P ?

- A. $\frac{3\pi}{2}$
 - B. $\frac{5\pi}{2}$
 - C. $\frac{\pi}{2}$
 - D. 2π
 - E. π
-

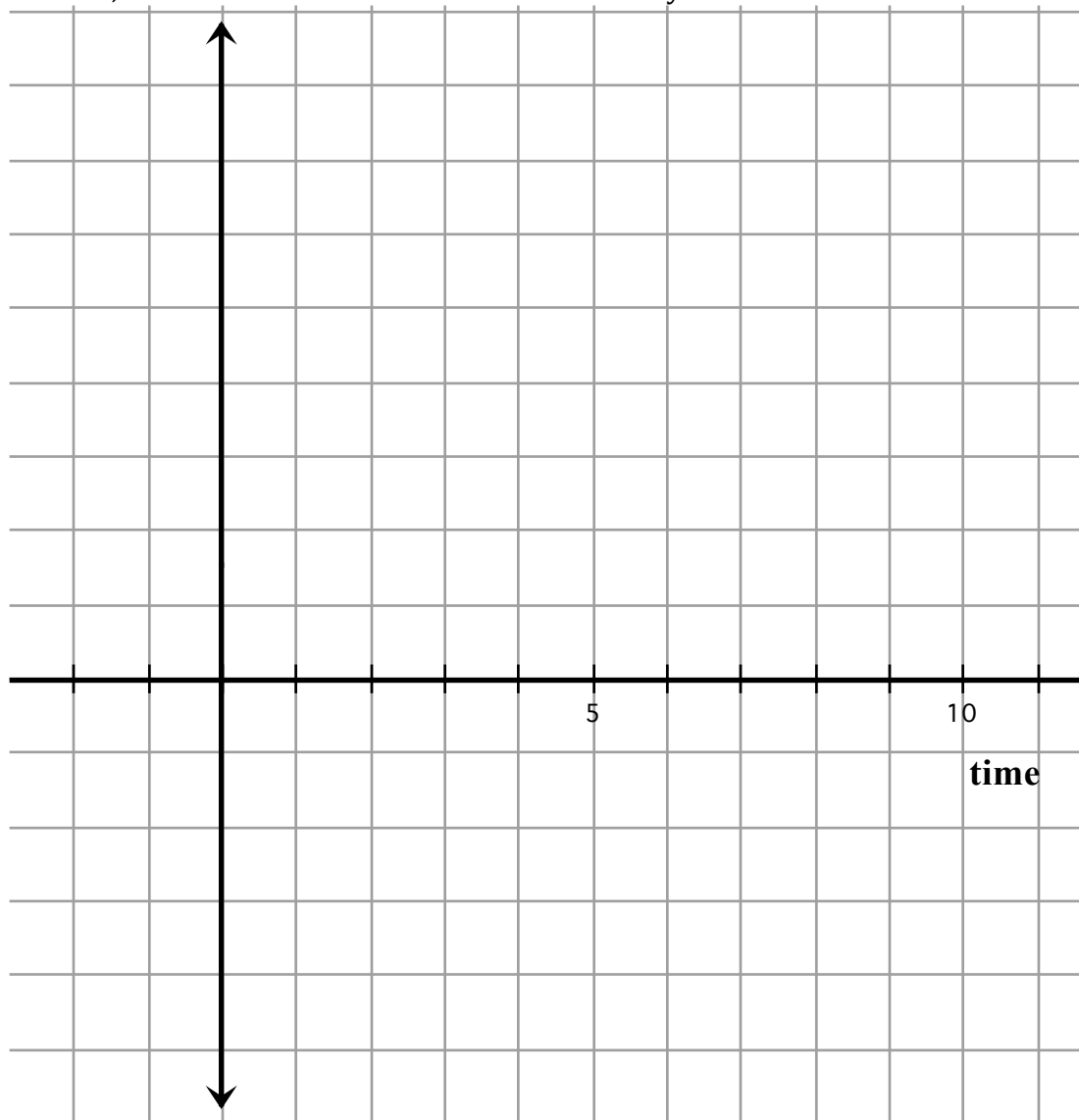


5B. In the right triangle above, $\sin(A) =$

- A. $5/3$
 - B. $4/5$
 - C. $3/4$
 - D. $4/3$
 - E. $3/5$
-

6B. Roselle is on a swing. The highest point above the ground that Roselle reaches is 9 ft, and the lowest point is 3 feet. It takes Roselle 2 seconds to travel that distance.

A. On the axes below, sketch a graph of Roselle's height off the ground as a function of time, for the first 10 seconds. Be sure to label your axes.



B. Write an equation that will model the height of Roselle off the ground over time.

APPENDIX E:

RUBRIC FOR SCORING PRE- AND POST-TEST ITEMS 6A AND 6B

4	The student sketched a graph with the appropriate period, maximum, and minimum values. The student defined a function that corresponded to the graph.
3	The student sketched a sinusoidal graph with the appropriate period, maximum, and minimum values. When defining the function, the student EITHER made an error computing the coefficient of the independent variable; OR switched the values of a and c in the function; OR used the wrong trigonometric function (only one of those errors).
2	<p>The student sketched a sinusoidal graph with the appropriate period, maximum, and minimum values. The student wrote a function to represent the graph, but the function contained at least two errors.</p> <p>OR</p> <p>The student sketched a graph, with an error in EITHER the amplitude, OR the period, OR the vertical shift. In addition to having one error in the graph, the student wrote a sinusoidal function that contained one error.</p>
1	The student has sketched a sinusoidal graph, with at least the correct maximum and minimum values. The student did not write a function to represent the graph, or the function has more than 2 errors.
0	<p>The student attempted to sketch a graph, but it did not have a sinusoidal shape. The student may have written an equation, but it did not define an appropriate function.</p> <p>OR</p> <p>The student left the problem blank.</p>

APPENDIX F:

POST-LESSON INTERVIEW PROTOCOL

As part of my school work, I study how students learn math. Right now I'm studying how kids learn with Etoys, and since you've been doing Etoys lessons, I want to know more about your experiences – what you liked, what you learned, and stuff like that – with this program.

So, I'm going to be asking you questions about how you learn math with Etoys. I'll be asking you about some things you did in Etoys lessons, what you liked or didn't like about Etoys, and about what you learned.

Are you okay with answering these questions now? Good, then let's get started.

Today is [DATE]. Could you please say your name?

Your answers are confidential, and they will not be shared with your teacher, parents, classmates, or anyone at the school. And I just want to remind you that we can stop the interview at any time. Also, when we use the transcripts in the future we will put fake names for the names of people and schools. When we're done, if you want to give me your fake name, I'll ask you if you have one or if you want me to pick one. Do you have any questions so far?

Students' perspectives about using the Etoys technology to solve the problem

1. Tell me about the Etoys lesson. What did you like about the lesson?
2. Were there things you didn't like about the lesson? What?
3. Did you like working with [PARTNER'S NAME]? Why or why not?
4. What's different about the Etoys lesson than what you usually do?

Students' thinking about modeling with trig functions after the Etoys lesson [Will vary based on the exact lesson]

Now I'm going to ask you to solve two problems similar to what you did on the Etoys lesson. You are allowed to use Etoys to solve the problems.

5. Describe what it means to do mathematical modeling.
6. Come up with a model for the scenario below. You can use Etoys to solve this problem if you want. [Give student a handout with the tides problem. Also, have an Etoys workbook opened. Allow the student *no more than 2 minutes of idle time and no more than 4(or 5) minutes if working but not moving to solution* to work on the problem.]
7. Describe to me what you did to come up with the model.
8. Can you think of any other way to model the scenario? If so, explain.
9. Another student used a sine function to model this problem [or cosine if the interview participant used a sine function]. Do you think that student's model could have been correct as well?

Students' perspectives about solving the problem with their partner

10. When you were working on the Etoys problem in class, were you the one who did most or all of the typing or was it your partner? Did you like that role? Why?

11. Did you and your partner agree about how to solve the problem? Were there any times when you were working on the problem that you two disagreed about something? Tell me about the disagreements.

12. That's it for my questions for you. Do you have any questions for me? Any other thoughts about the project that you want to ask or share with me?

Before we go, 2 things:

- 1) Did you have a name you want me to use on this interview instead of [REAL NAME]?
- 2) I need for you to not tell the other kids the questions because I need to see what THEY think, so PLEASE, will you promise for me, agree to wait until next [day you know you'll be done at that school] to talk about this with your friends?

THANKS!

APPENDIX G:

CODING CONVENTIONS FROM THE SYSTEM OF NEGOTIATION

	Code	Move	Description
Synoptic Moves			
	K1	Primary Knower	Provides information
	A1	Primary Actor	Performs an action
	K2	Secondary Knower	1. Asks a question 2. Suggests information to be confirmed by someone else
	A2	Secondary Actor	Makes a request for action
	dA1	Delayed Primary Actor	Offers to perform an action
	dK1	Delayed Primary Knower	Delays the provision of information
	K2f	Follow up by secondary knower	Follows up after a K1 move
	K1/K2 =	Elaboration move	Makes a restatement to a K1/K2 move
	K1/K2 +	Extension move	Adds some information to a K1/K2 move
	K1/K2 x	Enhancement move	Offers some condition to K1/K2 move
Dynamic Moves			
Suspending	cfrq	Request for confirmation	Requests confirmation that the previous utterance was heard correctly
	cf	Give confirmation	
	bch check	Backchannel Check	After making an utterance, the speaker checks to make sure that it was heard or understood.
Aborting	ch	Challenge	1. Challenges the validity of a prior statement 2. Defers a K1 move
	rch	Respond to challenge	Responds to case 1 of the Challenge moves described above
	hedge	Hedge	Hedges a K1 or K2 move (e.g., "He gets there at 8:00, but I don't really know.")
Elucidating	clfy	Clarification	Tries to clarify the meaning of a previous utterance
	rcly	Respond to clarification	

Sustaining	rp	Repeat	
	rph	Rephrase	
	corr	Correction	Corrects a K2 move
	self-corr	Self correction	Corrects a K1 or K2 moves (e.g., “He gets there at 8:00. No, I mean he gets there at 8:30.”)
	irr	Irrelevant response	
	ro	No response	
	rexp	Request for explanation	An individual in the position of K1 presses the K2 for more explanation
	exp	Explanation	Occurs in response to a request for explanation

Self-Correct versus Hedge: A self-corr move does something to revise, correct, or change the previous statement. A hedge move casts doubt about the previous statement, or expresses a degree of uncertainty, but does not provide any correction, revision, or alternative to the previous statement.

Response-to-Challenge versus initiating a new exchange: A response-to-challenge responds directly to the point made or the question raised in the challenge move. A new exchange could follow a challenge move if the person being challenge (a) establishes a new object of Negotiation or (b) changes positions.

K1-move versus Case 2 of a K2-move to initiate an exchange: This depends primarily on the voice inflection of a speaker. If the speaker offers some information, but raises the pitch of his or her voice at the end as though he or she is asking a question, this is classified as K2 move. If the speaker makes a declarative statement, it is a K1 move.

Determining when a new exchange has begun: There are 3 main factors that we use to distinguish between exchanges:

1. If a new object is being negotiated
2. The amount of time lapsed between moves. It’s possible that members of a group could have 2 consecutive exchanges about the same object of negotiation. If there is a lapse in conversation (more than would naturally occur in a continuous conversation between 2 people) that can indicate the start of a new exchange.
3. A change in positions. If a participant of an exchange switches from a K2 to a K1, or from a K1 to a K2, this indicates a new exchange. This is based on the assumption that an individual would not act as both an expert and a novice within the same exchange.

Determining when a new exchange has begun: If a speaker makes a sequence of challenge statements where one statement acts as the warrant for the other statement, then the two statements are part of a single challenge move complex (i.e., ch and ch+). If the two challenge statements are independent of one another, then they are two separate challenge moves (i.e., ch1 and ch2).

APPENDIX H:
TRANSCRIPTION CONVENTIONS

Convention	Meaning
-	A speaker interrupts his or her own turn without pausing.
()	Overlapping speech.
[]	Non-verbal actions, gestures, or pause in between speech.
:	Elongation by 1 second.
bold	Speech that illustrates the coding framework.